A CBA Pub Evening

- Next Wednesday, March 2, CBA arrange a pub evening
- We meet downtown, instructions will follow via email/studentportalen
- This event is non-mandatory and it is not a part of the course

Reading Instructions

Chapters for this lecture

- Chapter 5.1 – 5.2, and 5.4 – 5.8 in Gonzales-Woods.

Today’s lecture

- General concepts of image restoration (5.1 – 5.2 and 5.5 in GW)
- Periodic noise reduction by frequency domain filtering (5.4 in GW)
- Inverse and Wiener filtering (5.6 – 5.8 in GW)

Image Restoration

- Restore an image that has been degraded in some way.
- Make a model of the degeneration process and use inverse methods.
- Image restoration is an objective method using a priori information of the degradation.
- Image enhancing is a method to present the image in a “visually appealing” way.
- Image reconstruction is when you make an image from a large set of measurements or projections.

Model

- \( f(x, y) \) is the original image.
- \( H \) represents the system that affects our image.
- \( n(x, y) \) is disturbance, e.g., noise or external contribution.
- Obtained degraded image \( g(x, y) = H(f(x, y)) + n(x, y) \).
- Possible defects in the imaging system causing degradation:
  - Bad focusing.
  - Motion.
  - Non-linearity of the sensor.
  - Noise.
  - etc...
Image Restoration

Some possible approaches:
- Inverse filtering.
- Try to model degradation effect.
- Use Fourier-domain methods and identify which frequencies are related to the degrading effect.

Mathematical Fundamentals - Convolution

Convolution of Two Continuous Functions
\[ f(x, y) \otimes h(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta) h(x - \alpha, y - \beta) \, d\alpha \, d\beta \]

Convolution by the Impulse Function
\[ f(x, y) \otimes \delta(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta) \delta(x - \alpha, y - \beta) \, d\alpha \, d\beta = f(x, y) \]

Simplified Model

- Assume a model without noise \( n \).
  \[ g(x, y) = H[f(x, y)] \]
- Assume that the operator \( H \) is
  - linear: \( H[k_1 f_1(x, y) + k_2 f_2(x, y)] = k_1 H[f_1(x, y)] + k_2 H[f_2(x, y)] \)
  - additive: \( H[f_1(x, y)] + f_2(x, y) = H[f_1(x, y)] + H[f_2(x, y)] \)
  - homogeneous: \( H[k_1 f_1(x, y)] = k_1 H[f_1(x, y)] \)
  - position invariant: \( H[f(x - \alpha, y - \beta)] = g(x - \alpha, y - \beta) \)
- then, the following is true
  \[ g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta) H[\delta(x - \alpha, y - \beta)] \, d\alpha \, d\beta \]

Impulse Response

Impulse Response for Degradation Function \( H \)
\[ h(x, \alpha, \beta, \gamma) = H[\delta(x - \alpha, y - \beta)] \]
- In imaging systems the impulse response is called the point spread function (PSF).
- PSF describes how a point is imaged.

Fredholm Integral

Fredholm Integral or Superposition Integral of the First Kind
\[ g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta) h(x - \alpha, y - \beta) \, d\alpha \, d\beta \]
- The Fredholm integral says that if the impulse response is known, the response to any input signal can be calculated.
- If degradation function, \( H \), is position invariant, the integral is reduced to the convolution integral:

Fredholm Integral of Position Invariant Degradation Function
\[ g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta) h(x - \alpha, y - \beta) \, d\alpha \, d\beta \]

Point Spread Function

- Single impulse signal.
- Output blurred by the PSF.
- Two impulse signals.
- Output blurred by the PSF.
The (shift invariant) Model Revisited

In space: Convolution and addition
\[ g(x, y) = f(x, y) \otimes h(x, y) + n(x, y) \]

In frequency: Multiplication and addition
\[ G(u, v) = F(u, v) \cdot H(u, v) + N(u, v) \]

Fourier Methods

From Fourier Analysis:
- sinusoidal function in time (or spatial) domain of one variable corresponds to two impulse functions in Fourier domain.

Therefore,
- periodic noise in an image (i.e., repeated noise patterns) causes peaks in the FT of the image.
- by supressing these peaks and inverse transforming the image, a restored image is achieved.

Fourier Domain Filters
- Bandreject filters (ideal, Butterworth and Gaussian).

Notch filters (ideal, Butterworth and Gaussian).

Notch Filtering Example
- Original image.
- Fourier spectrum and it’s part removed with notch filter.
- Inverse transform of notch filtered image and result of notch filtering.

Inverse Filtering
- We can thus describe our model as \( g(x, y) = f(x, y) \otimes h(x, y) \).
- The Fourier transform gives
  \[ G(u, v) = F(u, v) \cdot H(u, v) \Rightarrow F(u, v) = \frac{G(u, v)}{H(u, v)} \]
- By modeling the degenerating effect \( h \) and dividing the FT of the image by the FT of the model, we can get the FT of a restored image.
- The inverse transform gives the restored image.
- The method is called inverse filtering.
Problems with Inverse Filtering

At deconvolution, the FT of the image is divided by the FT of the degrading effect.

- If noise is present, we get \( g(x, y) = f(x, y) \ast h(x, y) + n(x, y) \)
- \( \hat{F}(u, v) = \frac{G(u, v)}{H(u, v)} = \frac{F(u, v) + N(u, v)}{H(u, v)} \)

**Problems:**
- Small values of \( H(u, v) \) can cause overflow (usually small at HF).
- If noise is included, it can be dominating.

**Solutions:**
- Perform division only in a limited part of the \((u, v)\)-plane.
- Use weights to limit the effect at division with small numbers.

Inverse Filtering Examples

- Original image.
- Full filter. Radius of 70.
- Degraded image. Radius of 85.
- Radius of 40.

Norbert Wiener (1894 – 1964)


Wiener Filtering

- We consider images and noise as random processes.
- We try to find an estimate \( \hat{f} \) of the uncorrupted image \( f \) such that the mean square error is minimized:
  \[
  e^2 = E \left\{ (f - \hat{f})^2 \right\}
  \]

  If we assume the following conditions:
  - Noise and image are uncorrelated.
  - One or the other has zero mean.
  - Greylevels in the estimate are linear functions of degraded image.

  then the minimum of the error function is given by:
  \[
  \hat{F}(u, v) = \frac{H^*(u, v) S_f(u, v)}{S_f(u, v) |H(u, v)|^2 + S_n(u, v)} G(u, v)
  \]

  - \( K \) is adjusted interactively for best result.

Filter Comparison

(See the original and degraded image in previous slide.)

- Full inverse.
- Limited inverse.
- Wiener filter.

Figure 5.29 in GW – another comparison!
Drawback with Wiener Filtering

- Must know (or approximate) the power spectra of undegraded image and noise.
- We cannot deal with a spatially varying noise or PSF

A good thing:
- Even correlated noise can be removed

A possibly better choice:
- Constrained Least Squares Filter (CLSF).
  - Need only knowledge of mean and variance of noise (apart from degradation function).
  - Only one parameter, which can be iteratively computed for optimality.

Periodic noise reduction revisited

- How would you apply the Wiener filter to this $|G(u, v)|^2$?

$$
\hat{f}(u, v) = \left[ \frac{1}{|H(u, v)|^2 + \frac{\alpha}{\sigma(u, v)^2}} \right] G(u, v)
$$

Statistical Restoration

- Bayes theorem
  $$
P(f(x, y)|g(x, y)) = \frac{P(g(x, y)|f(x, y))P(f(x, y))}{P(g(x, y))}, \text{ or}
  
  P(f(x, y)) = \frac{P(g(x, y)|f(x, y))P(f(x, y))}{P(g(x, y))}
$$

- Find the most probable image, which generated $g(x, y)$
  $$
  \hat{f} = \arg \max_f P(g(x, y)|f(x, y))P(f(x, y)) = \arg \max_f P(g(x, y)|f(x, y))P(f(x, y))
  $$

- All linear theory (like the Wiener filter) is equivalent to a statistical approach based on Gaussian assumptions. The Wiener filter yields the most probable solution.

Restoration by Regularization

- Find $\hat{f}$ so that it looks like $f$ but is smooth ...
  $$
  \hat{f} = \arg \min_f \|g(x, y) - f(x, y)\|^2 + \alpha \|\nabla^2 f\|^2
  $$
  where $\nabla^2$ refers to the scalar product of the gradient operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

- We can use different norms too, for instance the $L_1$ (a.k.a. Total Variation denoising)
  $$
  \hat{f} = \arg \min_f \|g(x, y) - f(x, y)\|^2 + \alpha \|\nabla f\|^2
  $$

- Changing from $L_2$ to $L_1$ regularization can make a huge difference, edges are not punished as much for instance.
- Other things to penalize: Entropy of the histogram of $f$, penalize non-smooth functions, smooth functions.

Non-mandatory reading assignments

- Constrained Least Squares Filtering (CLSF) - section 5.9 in GW
- Mean Filters and order statistic filters in section 5.3 in GW
- Total Variation denoising: http://en.wikipedia.org/wiki/Total_variation_denoising
- Super resolution, e.g. http://people.csail.mit.edu/billf/superres/index.html