Previously:

- Solving $Ax = b$.
- Approximating derivatives with FD.
- Solving stationary PDEs with FD.
- Convergence analysis of $\nabla$:
  $$||u-u_n|| \leq ||A^n|| \||u||$$
  $\text{Truncation error} \Rightarrow \text{Difference operator specific}$
  $\text{Stability norm} \|A^n\|$  

- Lax-equivalence theorem
  $$(\text{consistency} + \text{stability} \iff \text{convergence})$$
  $(= \lim_{h \to 0} \text{when } h \to 0)$
  $(\|A^n\| \leq C)$
  $(C \text{ independent of } h)$

Today:

- Solving time-dependent PDEs with FD
- Von-Neumann stability criterion
Finite difference method for time-dependent PDEs

(Heat equation)

Find an unknown function \( u = u(x,t) \) s.t.

\[
\begin{align*}
\text{PDE} & \quad u_t = u_{xx} & \quad x \in (0,1), \quad t > 0 \\
\text{Dirichlet boundary conditions} & \quad u(0,t) = \alpha = 0 & \quad t > 0 \\
& \quad u(1,t) = \beta = 0 & \quad t > 0 \\
\text{Initial condition} & \quad u(x,0) = u_0(x)
\end{align*}
\]

\( u \) depends both on space and time. The FD solution procedure is different compared to the stationary PDE solution procedure.

Method of lines

- One of the solution procedures for time-dependent PDEs.

Idea

\[
\begin{align*}
\text{PDE} & \quad u_t(x,t) = u_{xx}(x,t) \\
\text{Solution} & \quad u_t(t) = DU(t)
\end{align*}
\]

Steps

1. Introduce a spatial grid:

2. Discretize the spatial operator in the interior:

\[
U_{xX}(x_i,t) \approx \frac{u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)}{h^2}
\]
This implies:

\[ u_t(x_i, t) = \frac{u(x_{i+1}, t) - 2u(x_i, t) + u(x_{i-1}, t)}{h^2} \quad i = 2, \ldots, N-1 \]

3. Impose boundary conditions

\[ u(x_1, t) = \alpha \]
\[ u(x_N, t) = \beta \]

\[ u_t(x_1, t) = \frac{u(x_{i+1}, t) - 2u(x_i, t) + \alpha}{h^2} \quad i = 2 \]
\[ u_t(x_i, t) = \frac{u(x_{i+1}, t) - 2u(x_i, t) + u(x_{i-1}, t)}{h^2} \quad i = 3, \ldots, N-2 \]
\[ u_t(x_N, t) = \frac{\beta - 2u(x_i, t) + u(x_{i-1}, t)}{h^2} \quad i = N-1 \]

4. Convert (3) to a matrix-vector form.

\[ \mathbf{u}_t(t) = \mathbf{D} \mathbf{u}(t), \quad \mathbf{u}_t(t|x_i) = \mathbf{D}_i \mathbf{u}(t) \quad i = 2, \ldots, N-1 \]

A system of ordinary differential equations (ODEs)

\[ \mathbf{u}_t = [u_t(x_1), u_t(x_2), \ldots, u_t(x_N)] \]
\[ \mathbf{u} = [u(x_1), u(x_2), \ldots, u(x_N)] \]

\[ \mathbf{D} \ldots \text{the fd matrix } \approx u_{xx}, \text{ with imposed BCs}. \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix} \]

5. Solve the system of ODEs in time: \[ t \to t_1, t_2, \ldots, t_m. \]

(Explicit/Implicit Euler, RK4, ...
Matlab example.

Observations:

a) When method stable, solution diffused in time \( \lim_{t \to \infty} u(t,x) = 0 \).

b) When method not stable, solution magnitude increased. \( \lim_{t \to \infty} u(t,x) = \infty \).

  a) is physically relevant, b) is not.

c) Stability depends on \( \Delta t \) and \( h \).

Question: When is the method stable?

- After we have already made a discretization in space:
  \[
  u_t = Du, 
  \]
  standard ODE stability criterion applies:

Example. Explicit Euler stability region.
Another perspective: Apply Forward Euler to

\[ u_1 = Du : \]

\[ \frac{u^{(2)}(x) - u^{(1)}}{\Delta t} = Du^{(1)} = u^{(1)} + \Delta t Du^{(1)} = (I + \Delta t D) u^{(1)} \]

We have

\[ u^{(1)} = (I + \Delta t D) u^{(1)} \]
\[ u^{(2)} = (I + \Delta t D) u^{(1)} = (I + \Delta t D)^2 u^{(1)} \]
\[ \vdots \]
\[ u^{(n)} = (I + \Delta t D)^{n-1} u^{(1)} \]

Therefore the solution \( u^n \) at \( t = t_2 \) is:

\[ u^n = (I + \Delta t D)^n u^{(1)} \]

initial condition.

\[ \Rightarrow \quad \| u^{(2)} \| = \| (I + \Delta t D)^2 u^{(1)} \| \]
\[ \leq \| (I + \Delta t D)^2 \| \| u^{(1)} \| \quad (\text{Cauchy-Schwarz ineq.}) \]
\[ \leq \| I + \Delta t D \|^{2-1} \| u^{(1)} \| \]

Thus: as \( q \to \infty \), \( \| u^{(2)} \| \) does not grow if:

\[ \| I + \Delta t D \| \leq 1 \]

amplification factor for explicit Euler.
Question. What relation should $\Delta t$ and $h$ have s.t. when the PDE operators approximated with:

$$u_{xx}(x_i, t_2) \approx \frac{u(x_{i+h}, t_2) - 2u(x_i, t_2) + u(x_{i-h}, t_2)}{h^2}$$

$$u_t(x_i, t_2) \approx \frac{u(x_i, t_{2+\Delta t}) - u(x_i, t_2)}{\Delta t},$$

yield stability when $u_t = u_{xx}$ is solved?

Von Neumann stability analysis

- based on Fourier analysis.
- for linear time-dependent PDEs, with constant coefficients.
- Periodic boundary conditions $\Rightarrow$ sufficient condition.
- Other boundary conditions $\Rightarrow$ only a necessary condition.
Example.

\[ \begin{align*}
    u_t &= u_{xx} & x \in (0, L) = \mathbb{R}, \ t > 0 \\
    u(0, t) &= u(L, t) &= t > 0 \\
    u_x(0, t) &= u_x(L, t) \\
    u(x, 0) &= u_0(x)
\end{align*} \]

Use \( u_{xx} \approx D_t^2 D_x u \) in space.
Use Forward Euler in time.

Write: \( u(x_j, t^n) = u^n_j \). Then we have:

\[ \frac{u_{j}^{n+1} - u_{j}^{n}}{\Delta t} = \frac{u_{j-1}^{n} - 2u_{j}^{n} + u_{j+1}^{n}}{h^2} \quad j = 1, \ldots, N \]

\[ u_{j}^{n} = u_{j}^{n+1} \Rightarrow \text{Periodicity} \]

Use ansatz: \( u_{j}^{n} = e^{i\xi_j h} \), \( u_{j}^{n+1} = Q(\xi) e^{i\xi_j h} \).

\( \xi \) ... constant.
\( Q(\xi) \) ... amplification factor.

Stability will depend on \( Q(\xi) \):

\[ u_{j}^{n+1} = Q(\xi) e^{i\xi_j h} = Q(\xi) u_{j}^{n} \Rightarrow |u_{j}^{n+1}| \leq |Q(\xi)| |u_{j}^{n}| \]

\[ \Rightarrow \quad |Q(\xi)| \leq 1 \quad \forall \xi, h, \Delta t \Rightarrow \text{Method unconditionally stable} \]
Goal. Based on $Q(\xi)$ find a condition for $\Delta t$ and $h$ s.t. the considered discretization becomes stable.

1. Insert the ansatz into the discretization.

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{h^2} \left( u_j^n - 2u_j^n + u_j^{n+1} \right)$$

$$Q(\xi) e^{i\xi j h} = e^{i\xi j h} + \frac{\Delta t}{h^2} \left( e^{i\xi (j-1)h} - 2e^{i\xi j h} + e^{i\xi (j+1)h} \right)$$

$$= e^{i\xi j h} + \frac{\Delta t}{h^2} \left( e^{i\xi j h} - e^{i\xi h} - 2e^{i\xi j h} + e^{i\xi h} e^{i\xi h} \right)$$

$$= e^{i\xi j h} + \frac{\Delta t}{h^2} e^{i\xi j h} \left( e^{-i\xi h} - 2 + e^{i\xi h} \right)$$

Now use $e^{-i\xi h} = \cos(\xi h) - i\sin(\xi h)$

$e^{i\xi h} = \cos(\xi h) + i\sin(\xi h)$

$$\Rightarrow Q(\xi) e^{i\xi j h} = e^{i\xi j h} \left[ 1 + \frac{\Delta t}{h^2} \left( 2\cos(\xi h) - 2 \right) \right]$$

$$\Rightarrow Q(\xi) = 1 + 2 \frac{\Delta t}{h^2} \left( \cos(\xi h) - 1 \right)$$
Since $-1 \leq \cos(\xi h) \leq 1$ and $\xi$ we have:

- $Q(\xi) = 1 + 2 \frac{\Delta t}{h^2} (\cos(\xi h) - 1)$
  
  $\leq 1 + 2 \frac{\Delta t}{h^2} (1 - 1) = 1$

- $Q(\xi) = 1 + 2 \frac{\Delta t}{h^2} (\cos(\xi h) - 1)$
  
  $\geq 1 + 2 \frac{\Delta t}{h^2} (-1 - 1) = 1 + 2 \frac{\Delta t}{h^2} (-2)$

$\Rightarrow \quad 1 + 2 \frac{\Delta t}{h^2} (-2) \leq Q(\xi) \leq 1$

We need $|Q(\xi)| \leq 1 \iff -1 \leq Q(\xi) \leq 1$

**Question**: When is

$$1 + 2 \frac{\Delta t}{h^2} (-2) \geq -1?$$

$\Rightarrow \quad 2 \frac{\Delta t}{h^2} (-2) \geq -2$

$\Rightarrow \quad 2 \frac{\Delta t}{h^2} \leq 1$

$\Rightarrow \quad \Delta t \leq \frac{h^2}{2}$

Thus: When $\Delta t \leq \frac{h^2}{2}$ we have $|Q(\xi)| \leq 1$ and the proposed method for $u_+ = u_{xx}$ is stable.