Lecture 8

Previously:  
- Adaptivity
- Stability of Forward Euler vs. Backward Euler

Quiz 5

Today:  
- Reiterate the stability of Backward Euler
- Stability regions for a scalar test $y$
- Stability regions for a system of test $y$
- ODE conclusion

Stability of Backward Euler

Recall: \[
\begin{cases}
  y' = & \lambda y \\
  y(0) = & y_0
\end{cases}
\quad \text{te } (0,T)
\]

Since the true solution is $y = y_0 e^{\lambda t}$, we know that for $\lambda < 0$, the solution decays.

![Graph showing stability of Backward Euler]

Question: For which $\lambda$ does Backward Euler admit this behaviour?

\[
\begin{align*}
  y_{i+1} & = y_i + \Delta t \lambda y_i \\
  \Rightarrow \quad y_{i+1} & = y_i + \Delta t \lambda y_i \\
  \Rightarrow \quad y_{i+1} & = y_i + \Delta t \lambda y_i \\
  \Rightarrow \quad y_{i+1} & = y_i \\
  \Rightarrow \quad y_{i+1} & = y_i \\
  \Rightarrow \quad \left| y_i \right| & \leq \left| \frac{1}{\lambda} \right|
\end{align*}
\]
Question: When is \( \left| \frac{1}{1-\alpha t} \right| \leq 1 \)

Observe: As long as the denominator \( |1-\alpha t| \geq 1 \), then
\[ \left| \frac{1}{1-\alpha t} \right| \leq 1. \]

Question: When is \( |1-\alpha t| \geq 1 \)?

a) \( 1-\alpha t \geq 1 \) \quad \text{or} \quad b) \( 1-\alpha t \leq -1 \)
\[ \Rightarrow -\alpha t \geq 0 \quad \Rightarrow -\alpha t \leq -2 \]
\[ \Rightarrow \alpha t \leq 0 \quad \quad \Rightarrow \alpha t \geq 2 \]

Observation: When \( \lambda < 0 \), then \( \alpha t \lambda \leq 0 \) is always true.
In this case, Backward Euler is unconditionally stable.
Stability regions for scalar test eq.

\[ \begin{align*}
  y' &= \lambda y & t \in (0, T] \\
  y(0) &= y_0
\end{align*} \]

Recall Forward Euler stability condition:

\[ \Delta t \lambda \leq 0 \quad \text{and} \quad \Delta t \lambda \geq -2 \]

We can draw this condition. \( \lambda \in \mathbb{R} \)

Not stable \(-2\) stable \(0\) Not stable \(\Delta t \lambda\)

Stability only for \(\lambda < 0\).

Example. Let \(\lambda = -0.001\). Then \(\Delta t \lambda \geq -2\) and \(\Delta t \lambda \leq 0\) if \(\Delta t + \frac{2}{1000} = -0.002\).
Recall Backward Euler stability condition.

\[ \Delta t \lambda \leq 0 \text{ or } \Delta t \lambda \geq 2 \]

If we draw it, we get:

![Stability diagram](image)

(1) and (2) are stability regions for FE and BE respectively.

**Question:** But what if \( \lambda \) is a complex number? Let's consider an example: \( \lambda = -100 + 100i \).

Similarly as in Lecture 7, one can derive that:

**Forward Euler:** \( |1 + \Delta t \lambda| \leq 1 \) gives

\[ \text{Re}(\Delta t \lambda) \geq -2 \quad \text{Re}(\Delta t \lambda) \leq 0 \quad \text{Im}(\Delta t \lambda) \leq 4i \]

**Backward Euler:** \( \Delta t \lambda \) gives

\[ \text{Re}(\Delta t \lambda) \geq 2 \quad \text{Im}(\Delta t \lambda) \leq 4i \]

**Observe:** The condition on the real axis is the same as for \( \lambda \in \mathbb{R} \).
Backward Euler: \[ \left| \frac{1}{1 - \lambda h} \right| \leq 1 \] gives

\[ \text{Re}(\lambda h) \geq 0 \quad \text{Im}(\lambda h) \geq 1i \]
\[ \text{Re}(\lambda h) \leq -2 \quad \text{Im}(\lambda h) \leq -1i \]

**Observations:**

\[ \text{Re}(\lambda) \text{ allows smaller Re}(\lambda h) \text{ compared with FE.} \]

If \( \text{Re}(\lambda) = 0 \), then FE unstable, but RK4 and it can be stable.

When \( \text{Re}(\lambda) \) very small, then ODE is stiff.

**Question:** What about the stability regions of other ODE methods for solving \( y' = \lambda y \)?

\[ \Rightarrow \text{It turns out that every method has its own stability region.} \]

\[ \Rightarrow \text{Roughly speaking, higher we go in the order of the method, more forgiving the stability region.} \]

**Observations:**

- RK4 allows smaller Re(\( \lambda h \)) compared with FE.
- If Re(\( \lambda \)) = 0, then FE unstable, but RK4 and it can be stable.
- When Re(\( \lambda \)) very small, then ODE is stiff.
Stability regions for systems of test ODEs

Consider a 2nd order ODE:
\[
\begin{cases}
y''(t) = y(t) + \epsilon \in (0, T] \\
y(0) = y_0 \\
y'(0) = y_0'
\end{cases}
\]

which we can rewrite into a system of 1st order ODEs:
\[
\begin{align*}
u_1 &= y \\
u_1' &= y' = u_2 \\
u_2 &= y'' = y = u_1 \\
u_2' &= y' = u_1
\end{align*}
\]

Write this to a matrix-vector form.
\[
\begin{pmatrix}
u_1' \\
u_2'
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix},\quad u_1(0) = y_0, \quad u_2(0) = y_0'
\]

\[
\begin{pmatrix}
u_1' \\
u_2'
\end{pmatrix} = D
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix}
\]

Statement 1:

All eigenvalues \( \lambda \in \mathbb{C} \) of a matrix \( D \) have to fall into the FE stability region so that the formula
\[
u_{i+1} = \nu_i + \alpha + D \nu_i
\]
is stable.
An overview

ODEs:
\[
\begin{align*}
  y' &= f(t, y(t)) \quad t \in (0, T) \\
  y(0) &= y_0
\end{align*}
\]

BVP

An ODE of order \( k \) can be converted to a system of 1st order ODEs of size \( k \).

Numerical methods for ODEs.

**Explicit**
- Forward Euler \( O(\Delta t) \)
- Heun \( O(\Delta t^2) \)
- RK4 \( O(\Delta t^4) \)

- Can be unstable wrt \( \Delta t \)
- Easy to apply
- Good for non-stiff ODEs
- ODE45

**Implicit**
- Backward Euler \( O(\Delta t) \)

- Unconditionally stable wrt \( \Delta t \)
- For every step, we need to solve an equation
- Good for stiff ODEs
- ODE45
Adaptivity

Fix an error tolerance $\text{tol}$. Then $\Delta t$ in every step is chosen such that $|\tilde{e}_i| \leq \text{tol}$, where $\tilde{e}$ is an approximation to the error in every step.

ODE45: stepping with RK4, $\Delta t$ based on $\tilde{e}_i \approx |y_{i+1} - y_i|$.