Image restoration
Lecture 15

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Today’s lecture

- General concepts of image restoration (5.1 – 5.2 and 5.5 in GW)
- Periodic noise reduction by frequency domain filtering (5.4 in GW)
- Inverse and Wiener filtering (5.6 – 5.8 in GW)
The Fast Fourier Transform (FFT)

Discrete Fourier Transform (DFT)

\[ F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi \frac{ux}{M}} \text{ for } u = 0, 1, \ldots, M - 1 \]

\[ F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi \left( \frac{ux}{M} + \frac{vy}{N} \right)} \text{ and } v = 0, 1, \ldots, N - 1 \]

Fast Fourier Transform is an efficient algorithm for computing the DFT.

For a 1-D signal:

- The ordinary DFT of \( M \) points require \( M^2 \) operations (multiplications/additions).
- The FFT of \( M \) points require \( M \log_2 M \) operations.

Implementation of FFT in 2D is analogous to the 1D case as 2D-DFT is separable transformation.
Filtering in the Frequency Domain

The basics

1. Multiply the input image by \((-1)^{x+y}\) to center the transform.
2. Compute \(F(u, v)\), the DFT of the image from (1).
3. Multiply \(F(u, v)\) by a filter function \(H(u, v)\).
4. Compute the inverse DFT of the result in (3).
5. Obtain the real part of the result in (4).
6. Multiply the result in (5) by \((-1)^{x+y}\).

\[
G(u, v) = H(u, v)F(u, v)
\]

Element-by-element multiplication.
In the frequency domain, define a cut-off radius $D_0$.

**Ideal Lowpass Filter (ILPF)**

$$H(u, v) = \begin{cases} 
1 & \text{if } D(u, v) \leq D_0; \\
0 & \text{if } D(u, v) > D_0.
\end{cases}$$

To find $h(x, y)$:

1. **Centering**: $H(u, v) \ast (-1)^{u+v}$
2. **Inverse Fourier transform**
3. **Multiply real part by** $(-1)^{x+y}$
Ideal Lowpass Filters and "Ringing"

Properties of $h(x, y)$:

1. It has a central dominant circular component (providing the blurring)
2. It has concentric circular components (rings) giving rise to the ringing effect.

Example of $h(x)$ from the inverse FT of a disc (ILPF) with radius 5.
Ideal Lowpass Filters and "Ringing"

Original image (top left) and filtered images with ILPF of radius 5, 15 and 30, removing 8, 5.4 and 3.6% of the total power. This type of artefacts are not acceptable in e.g. medical imaging.
Butterworth Lowpass Filter (BLPF)

\[
H(u, v) = \frac{1}{1 + \left( \frac{D(u,v)}{D_0} \right)^{2n}}
\]

- \(n\) is the order of the filter
- A high \(n\) will cause "ringing" (approaching ILPF)
- No sharp discontinuity
Butterworth Lowpass Filter

In general, BLPFs of order 2 are a good compromise between effective lowpass filtering and acceptable ringing characteristics.
Gaussian Lowpass Filter

Gaussian Lowpass Filter (GLPF)

\[ H(u, v) = e^{-\frac{D^2(u, v)}{2D_0^2}} \]

- \( D_0 \) is the standard deviation (\( \sigma \)), or the "spread" of the Gaussian.
- The inverse FT of a Gaussian is also a Gaussian, meaning a Gaussian smoothing in the spatial domain.
- Guarantees no ringing.

\[ H(u, v) \]

\[ D(u, v) \]
Highpass Filters

Ideal Highpass Filter (IHPF)

\[ H(u, v) = \begin{cases} 
0 & \text{if } D(u, v) \leq D_0; \\
1 & \text{if } D(u, v) > D_0.
\end{cases} \]

Butterworth Highpass Filter (BHPF)

\[ H(u, v) = 1 - \frac{1}{1 + \left( \frac{D(u, v)}{D_0} \right)^{2n}} = \frac{1}{1 + \left( \frac{D_0}{D(u, v)} \right)^{2n}} \]

Gaussian Highpass Filter (GHPF)

\[ H(u, v) = 1 - e^{-\frac{D^2(u, v)}{2D_0^2}} \]

- Same ringing effect as for the lowpass filters.
Highpass Filters

Spatial representation of different highpass filters (ideal, Butterworth and Gaussian).

Ringing effect noticeable in ideal and Butterworth filters.
High-Frequency Emphasis Filtering

\[ H_{\text{hfe}}(u, v) = a + b \ast H_{\text{hp}}(u, v) \]

Result of high-frequency emphasis filtering (original, highpass; high-freq emphasis, histogram equalization).
Image Restoration

- Restore an image that has been degraded in some way.
- Make a model of the degeneration process and use inverse methods.
- **Image restoration** is an objective method using a priori information of the degradation.
- **Image enhancing** is a method to present the image in a "visually appealing" way.
Model

\[ f(x, y) \] is the original image.

\[ H \] represents the system that affects our image.

\[ n(x, y) \] is disturbance, e.g., noise or external contribution.

Obtained degraded image \[ g(x, y) = H(f(x, y)) + n(x, y) \].

Possible defects in the imaging system causing degradation:

- Bad focusing.
- Motion.
- Non-linearity of the sensor.
- Noise.
- etc...
Image Restoration

Possible approaches:

- Inverse filtering.
- Try to model degradation effect.
- Use Fourier-domain methods and identify which frequencies are related to the degrading effect.
Mathematical Fundamentals - Convolution

Convolution of Two Continuous Functions

\[ f(x, y) \otimes h(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta) h(x - \alpha, y - \beta) \, d\alpha \, d\beta \]

Convolution by the Impulse Function

\[ \int_{-\infty}^{\infty} f(x) \delta(x - x_0) \, dx = f(x_0) \]

Note: The sifting property (L13) can be represented as convolving with the impulse function.
Simplified Model

Assume a model without noise \( n \).

\[
g(x, y) = H[f(x, y)]
\]

Assume that the operator \( H \) is

- linear: \( H[k_1f_1(x, y) + k_2f_2(x, y)] = k_1H[f_1(x, y)] + k_2H[f_2(x, y)] \)
- additive: \( H[f_1(x, y) + f_2(x, y)] = H[f_1(x, y)] + H[f_2(x, y)] \)
- homogeneous: \( H[k_1f_1(x, y)] = k_1H[f_1(x, y)] \)
- position invariant: \( H[f(x - \alpha, y - \beta)] = g(x - \alpha, y - \beta) \)

then, the following is true

\[
g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta) H[\delta(x - \alpha, y - \beta)] d\alpha d\beta
\]
Impulse Response

Impulse Response for Degradation Function $H$

$h(x, \alpha, y, \beta) = H[\delta(x - \alpha, y - \beta)]$

- In imaging systems the impulse response is called the point spread function (PSF).
- PSF describes how a point is imaged.
Fredholm Integral

Fredholm Integral or Superposition Integral of the First Kind

\[
g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta) h(x, \alpha, y, \beta) \, d\alpha \, d\beta
\]

- The Fredholm integral says that if the impulse response is known, the response to any input signal can be calculated.

- If degradation function, \(H\), is position invariant, the integral is reduced to the convolution integral:

Fredholm Integral of Position Invariant Degradation Function

\[
g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta) h(x - \alpha, y - \beta) \, d\alpha \, d\beta
\]
Point Spread Function

- Single impulse signal.
- Output blurred by the PSF.

- Two impulse signals.
- Output blurred by the PSF.
Fourier Methods

From Fourier Analysis:

A sinusoidal function in time (or spatial) domain of one variable corresponds to two impulse functions in Fourier domain.

Therefore,

- periodic noise in an image (i.e., repeated noise patterns) causes peaks in the FT of the image.
- by suppressing these peaks and inverse transforming the image, a restored image is achieved.

(The last problem in computer exercise 1)
Fourier Domain Filters

- Bandreject filters (ideal, Butterworth and Gaussian).

- Notch filters (ideal, Butterworth and Gaussian).
Periodic Noise Reduction
Notch Filtering Example

- Original image.

- Fourier spectrum and its part removed with notch filter.

- Inverse transform of notch filtered image and result of notch filtering.
Inverse Filtering

- We can thus describe our model as $g(x, y) = f(x, y) \otimes h(x, y)$.
- The Fourier transform gives

$$G(u, v) = F(u, v) \cdot H(u, v) \iff \hat{F}(u, v) = \frac{G(u, v)}{H(u, v)}$$

- By modeling the degenerating effect ($h$) and dividing the FT of the image by the FT of the model, we can get the FT of a restored image.
- If noise is present, we get $g(x, y) = f(x, y) \otimes h(x, y) + n(x, y)$ and

$$\hat{F}(u, v) = \frac{G(u, v)}{H(u, v)} - \frac{N(u, v)}{H(u, v)}$$

- The inverse transform gives the restored image.
- The method is called inverse filtering.
Problems with Inverse Filtering

At deconvolution, the FT of the image is divided by the FT of the degrading effect.

- **Problems:**
  - Small values of $H(u, v)$ can cause overflow (usually small at HF).
  - If noise is included, it can be dominating.

- **Solutions:**
  - Perform division only in a limited part of the $(u, v)$-plane.
  - Use weights to limit the effect at division with small numbers.
Inverse Filtering Examples

- **Original image.**
- **Full filter.**
- **Radius of 70.**

- **Degraded image.**
- **Radius of 85.**
- **Radius of 40.**
Wiener Filtering

- We consider images and noise as random processes.
- We try to find an estimate \( \hat{f} \) of the uncorrupted image \( f \) such that the mean square error is minimized.

\[
e^2 = E \left\{ (f - \hat{f})^2 \right\}
\]

If we assume the following conditions:
1. Noise and image are uncorrelated.
2. One or the other has zero mean.
3. Greylevels in the estimate are linear functions of the graylevels in the degraded image.

...then the minimum of the error function (in the frequency domain) is given by

\[
\hat{F}(u, v) = \left[ \frac{H^* (u, v) S_f (u, v)}{S_f (u, v) |H (u, v)|^2 + S_\eta (u, v)} \right] G (u, v)
\]
The Wiener filter can be written in a simpler form

\[ \hat{F}(u, v) = \frac{1}{H(u, v)^2 + \frac{S_\eta(u, v)}{S_f(u, v)}} \left( \frac{|H(u, v)|^2}{H(u, v)^2 + \frac{S_\eta(u, v)}{S_f(u, v)}} \right) G(u, v) \]

Note that if the noise is zero, the Wiener filter reduces to the inverse filter.

The noise power spectrum is constant when dealing with spectrally white noise.

Approximation frequently used

\[ \hat{F}(u, v) = \frac{1}{H(u, v)^2 + K} \left( \frac{|H(u, v)|^2}{H(u, v)^2 + K} \right) G(u, v) \]

\( K \) is adjusted interactively for best result.
Filter Comparison

See the original and degraded image in slide 30.

- Full inverse.
- Limited inverse.
- Wiener filter.

Figure 5.29 - another comparison!
Drawback with Wiener Filtering

Drawbacks:
- Must know (or approximate) the power spectra of undegraded image and noise.
- Spatially invariant degradations cannot be restored.

Better choice:
- Constrained Least Squares Filter (CLSF).
  - Need only knowledge of mean and variance of noise (apart from degradation function).
  - Only one parameter, which can be iteratively computed for optimality.
Non-mandatory reading assignments

- Constrained Least Squares Fitting (CLSF) - section 5.9 in GW
- Deblurring Using a General Linear Models - handouts
- The Wavelet Tutorial -
  [http://users.rowan.edu/~polikar/WAVELETS/WTtutorial.html](http://users.rowan.edu/~polikar/WAVELETS/WTtutorial.html)
- Wavelets and Multiresolution Processing - chapter 7 in GW
Chapter 8 in Gonzales-Woods.