Reading Instructions

Chapters for this lecture

- Chapter 4.1 and 4.7 – 4.10 in Gonzales-Woods. (4.2-4.6)

Previous Lecture

Filtering

Subjects

- Neighborhood relationships (adjacency, connectivity).
- Linear filtering (Averaging).
- Non-linear filtering (Min, Max, and Median).
- Edge filters (Sobel, Laplace, and edge enhancing).
The Fourier Transform
Jean Baptiste Joseph Fourier (March 21, 1768 - May 16, 1830)

- French mathematician (“Théorie analytique de la chaleur”, 1822).
- Fourier series: periodically repeating functions can be expressed as a sum of sines and/or cosines of different frequencies and weights.
- Fourier transform: Functions that are not periodic (but with finite area under the curve) can be expressed as the integral of sines and/or cosines of different frequencies and weights.
- Representation using Fourier series or transforms allows for complete recovery of the original function.
- Different way to represent the same information.
- Light & prism: Separation of wavelengths.

The function at the bottom is the sum of the four functions above it. Fourier’s idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

The Fourier Transform
Continuous 1D and 2D case

The one-dimensional Fourier transform

\[ F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} \, dx \]

and its inverse

\[ f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} \, du. \]

Fourier transform pair in 2D,

\[ F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi (ux + vy)} \, dx \, dy \]

and,

\[ f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi (ux + vy)} \, du \, dv. \]

The Fourier Transform
Discrete 1D case

Discrete Fourier transform

\[ F(u) = \frac{1}{M} \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \text{ for } u = 0, 1, ..., M - 1. \]

- Euler’s formula \( e^{j\theta} = \cos(\theta) + j\sin(\theta) \).

- Substitution gives

\[ F(u) = \frac{1}{M} \sum_{x=0}^{M-1} f(x) \left[ \cos \left( \frac{2\pi ux}{M} \right) - j\sin \left( \frac{2\pi ux}{M} \right) \right] \text{ for } u = 0, 1, ..., M - 1. \]

For each term of the Fourier transform \( \{F(u)\} \) is made up of all values of the function \( f(x) \).

The discrete Fourier transform always exist!
The Fourier Transform

Discrete 1D case

Express \( F(u) \) in polar coordinates

\[
F(u) = |F(u)| e^{-i\Theta(u)}
\]

where the magnitude (spectrum) of the Fourier spectrum is

\[
|F(u)| = \left[ R^2(u) + I^2(u) \right]^{1/2}
\]

and the phase angle (phase spectrum) is

\[
\Theta(u) = \tan^{-1}\left( \frac{I(u)}{R(u)} \right).
\]

The power spectrum (spectral density) is

\[
P(u) = |F(u)|^2 = R(u)^2 + I(u)^2.
\]

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Introduction to the Frequency Domain

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The Fourier Transform

Properties of the FT

- Separable: The 2D Fourier transform can be performed by two 1D transforms

\[
f(x, y) \xrightarrow{row} F(x, v) \xrightarrow{column} F(u, v).
\]

- Linear:

\[
\mathcal{F}(af(x) + bg(x)) = aF(u) + bG(u).
\]

- Translation:

\[
\mathcal{F}(f(x - x_0)) = F(u)e^{-2\pi iux_0/M} \quad \mathcal{F}(f(x)e^{2\pi iux_0/M}) = F(u - u_0).
\]

- Rotation: Polar coordinates. The angles are the same in spatial and frequency domain.

\[
x = r \cos(\theta) \quad u = w \cos(\phi)
y = r \sin(\theta) \quad v = w \sin(\phi)
\]

\[
\mathcal{F}(f(r, \theta + \theta_0)) = F(w, \phi + \theta_0)
\]

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Introduction to the Frequency Domain

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The Fourier transform
GW example 4.1: Implemented in MATLAB

\[ M = 1024; \]
\[ A = 1; \]
\[ K = 8; \]
\[ f = \text{zeros}(M, 1); \]
\[ f(1:K) = \text{ones}(K, 1); \]
\[ R = \text{zeros}(M, 1); \]
\[ I = \text{zeros}(M, 1); \]
\[ \text{for } u = 0:M-1 \]
\[ \quad \text{for } x = 0:M-1 \]
\[ \quad \quad R(u+1) = R(u+1) + ((-1)^x + f(x+1) \cos(2 \pi u x / M)); \]
\[ \quad \quad I(u+1) = I(u+1) + ((-1)^x + f(x+1) \sin(2 \pi u x / M)); \]
\[ \quad \text{end} \]
\[ \quad F(u+1) = 1/M \times \sqrt{R(u+1) \times R(u+1) + I(u+1) \times I(u+1)}; \]
\[ \text{end} \]

The Fourier Transform
1D Discrete fourier Transform

1D image \((x = 0..1024)\)
1D Fourier transform \((u = 0..1024)\)

The Fourier Transform
Inverse 1D DFT

\[ \text{invF} = \text{zeros}(M, 1); \]
\[ \text{for } x = 0:M-1 \]
\[ \quad \text{invF}(x+1) = \text{invF}(x+1) + R(u+1) \times \cos(2 \pi u x / M); \]
\[ \quad \text{invF}(x+1) = \text{invF}(x+1) + I(u+1) \times \sin(2 \pi u x / M); \]
\[ \quad \text{end} \]
\[ \quad \text{invF}(x+1) = (-1)^x \times (1/M) \times \text{invF}(x+1); \]
\[ \text{end} \]

The Fourier Transform
GW example 4.2
left Image of a \(20 \times 40\) white rectangle on a black background of size \(512 \times 512\) pixels.
right Centered Fourier spectrum shown after application of the log transformation
\[ D(x, y) = c \log(1 + |F(u, v)|). \]
The Fourier Transform

Original. Low frequencies. High frequencies.

The Fourier Transform

Rotation

\[ f(x, y) \]

\[ F(f(x, y)) \]
The Fourier Transform
Filtering in the frequency domain

Basic idea:
1. Fourier transform an image.
2. Manipulate the transform by suppressing certain parts.
3. Inverse transform and get a new “better” image.

Most common is to use a filter which affects the real and the imaginary part of the transform equally, i.e., the phase is not changed (“zero-phase shift filters”).

The Fourier Transform
Convolution (swedish “faltning”)

- Convolution of two continuous 1D functions $f(x)$ and $h(x)$ is defined as
  \[ f(x) * h(x) = \int_{-\infty}^{+\infty} f(\alpha)h(x-\alpha) \, d\alpha. \]
- In 2D we get
  \[ f(x, y) * h(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\alpha, \beta)h(x-\alpha, y-\beta) \, d\alpha \, d\beta. \]
- Convolution of two discrete functions $f(x, y)$ and $h(x, y)$
  \[ f(x, y) * h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n)h(x-m, y-n). \]

The Convolution theorem
States that
\[ f(x, y) * h(x, y) \Leftrightarrow F(u, v) \cdot H(u, v) \]
and
\[ f(x, y) \cdot h(x, y) \Leftrightarrow F(u, v) * H(u, v), \]
i.e., convolution in the spatial domain corresponds to multiplication in the frequency domain, and vice versa.

The convolution theorem makes up the fundamental link between image processing in the spatial domain and the frequency domain.
The Fourier Transform

Low pass filters

- We “remove” high frequencies. This corresponds to a smoothing filter in the spatial domain.
- The simplest case is to use an ideal lowpass filter (ILPF) where $H(u, v)$ is set to 0 outside a given frequency.

$$H(u, v) = \begin{cases} 
1 & D(u, v) \leq D_0 \\
0 & D(u, v) > D_0 
\end{cases}$$

- Problem: ILPF results in “rings” in the image. These artefacts are avoided by not using a sharp cut-off frequency. A smooth cut-off gives a nicer result (e.g., Butterworth, Gauss, etc.).

Highpass filters

- We “remove” low frequencies. This corresponds to an edge enhancing filter in spatial domain.
- The simplest case is to use an ideal highpass filter (IHPF) where $H(u, v)$ is set to 0 inside a given frequency.

$$H(u, v) = \begin{cases} 
0 & D(u, v) \leq D_0 \\
1 & D(u, v) > D_0 
\end{cases}$$

- Problem: IHPF will just like ILPF result in “rings” in the image. A smooth cut-off gives a nicer result (e.g., Butterworth, Gauss, etc.).
The Fourier Transform

Other filters

**Bandpass:** Allow frequencies in a band between two frequencies $D_0$ and $D_1$.

**Bandstop:** Stops frequencies in a band between two frequencies $D_0$ and $D_1$.

**Other:** Filters which allow different frequencies in the $u$, and $v$ direction.

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**Conclusions**

- The Fourier transform and the convolution theorem make up the fundamentals of image processing in the frequency domain.
- Filtering in the frequency domain gives the same results as linear filtering in the spatial domain.
- Why filter in the frequency domain?
  - Save time.
  - Opens the possibilities of image processing which is simply not possible in the spatial domain (see lecture on FFT).

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**Example**

$f(x, y)$

$F(f(x, y))$
The Fourier Transform

More examples

Reading Instructions

Chapters next lecture

- Chapter 10.1 – 10.2.5 and 10.3 – 10.5 in Gonzales-Woods.