1. (a) The equilibrium points are given by $\dot{x} = 0$. There are three separate cases that must be examined.

$x_1 > 1$:
The control signal is $u = -2$ which gives the system

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

Setting $\dot{x} = 0$ gives an equilibrium point in $x_0 = [2 \ -2]^T$.

$|x_1| \leq 1$:
The control signal is $u = -2x_1$ which gives the system

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} -2x_1 \\ -2x_1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix} x$$

Setting $\dot{x} = 0$ gives an equilibrium point in $x_0 = [0 \ 0]^T$.

$x_1 < -1$:
The control signal is $u = 2$ which gives the system

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Setting $\dot{x} = 0$ gives an equilibrium point in $x_0 = [-2 \ 2]^T$.

(b) A system $\dot{x} = f(x)$ can be linearised to $\dot{x} = Ax$ around the equilibrium points $x_0$, where $A$ is the jacobian $\frac{\partial f}{\partial x}|_{x=x_0}$. The eigenvalues $\lambda$ for a matrix $A$ are given by $\det(A - \lambda I) = 0$ and the eigenvectors $v$ are given by $(A - \lambda I)v = 0$.

$x_1 > 1$:
From a) we can write the system as

$$f(x) = \begin{cases} 
\dot{x}_1 = x_1 - 2 \\
\dot{x}_2 = -x_2 - 2
\end{cases}$$

The jacobian is

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The eigenvalues of $A$ are $\lambda_1 = 1$ and $\lambda_2 = -1$. The eigenvectors are

$$(A - \lambda_1 I)v_1 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} v_1 \rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(A - \lambda_2 I)v_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} v_2 \rightarrow v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
This equilibrium point is a saddle point since the eigenvalues have different signs.

\(|x_1| \leq 1:\)

From a) it is clear that the system already is linear with

\[ A = \begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix} \]

The eigenvalues of A are \( \lambda_1 = \lambda_2 = -1 \). The eigenvector is

\[ (A - \lambda_1 I)v_1 = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} v_1 \rightarrow v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

This equilibrium point is a stable node (entangentnod) since the eigenvalues are negative and equal with one common eigenvector.

\( x_1 < -1:\)

From a) we can write the system as

\[ f(x) = \begin{cases} \dot{x}_1 = x_1 + 2 \\ \dot{x}_2 = -x_2 + 2 \end{cases} \]

The jacobian is

\[ A = \frac{\partial f}{\partial x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]

This linearized system is the same as for \( x_1 > 1 \), hence this is a saddle point.

(c) The phase plot should include: The equilibrium points, the directions of the eigenvectors at the equilibrium points and arrows which indicate the behaviour of the solution between the equilibrium points.
2. (a)

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = -\frac{1}{m} \left( b x_2 |x_2| + k_0 (x_1 - s_0) + k_1 (x_1 - s_0)^3 \right) \]

(b) 
\[
\begin{cases}
\dot{x}_1 = 0 \\
\dot{x}_2 = 0
\end{cases} \Rightarrow 
\begin{cases}
x_2 = 0, \\
\frac{1}{m} (x_1 - s_0) (k_0 + k_1 (x_1 - s_0)^2) = 0
\end{cases} \Rightarrow
\begin{cases}
x_2 = 0, \\
x_1 = s_0
\end{cases}
\]

Hence the only equilibrium point \( x_0 \) is \( x_1 = s_0, x_2 = 0 \) (which corresponds to our physical intuition).

(c) The derivative of \(|x_2|\) is undefined at \( x_2 = 0 \), and a linearization around this (equilibrium) point is therefore not possible.

(d) If we use the total energy in the system as a Lyapunov function, we have
\[ V(x) = \frac{m x_2^2}{2} + k_0 \frac{(x_1 - s_0)^2}{2} + k_1 \frac{(x_1 - s_0)^4}{4}. \]

We start by confirming that (12.4) and (12.5b) holds for this function:
\[ V(x) = \frac{m 0^2}{2} + k_0 \frac{(s_0 - s_0)^2}{2} + k_1 \frac{(s_0 - s_0)^4}{4} = 0 \]
\[ V(x) > 0 \text{ for } x \neq x_0, \text{ since } m x_2^2 > 0, k_0 (x_1 - s_0)^2 > 0, k_2 (x_1 - s_0)^4 > 0 \]
\[ V_2(x) f(x) = \left[ k_0 (x_1 - s_0) + k_1 (x_1 - s_0)^3 \right] m x_2 \left[ -\frac{1}{m} (b x_2 |x_2| + k_0 (x_1 - s_0) + k_1 (x_1 - s_0)^3) \right] \]
\[ = -b x_2^2 |x_2| \leq 0 \]
\[ V(x) \to \infty \text{ as } |x| \to \infty, \text{ since } |x| \to \infty \text{ implies that } |x_1| \to \infty \text{ or } |x_2| \to \infty \text{ (or both), implying that } V(x) \to \infty. \]

We note, however, that \( V_2(x) f(x) = 0 \) for certain \( x \neq x_0 \) (e.g., \( x_1 = -1, x_2 = 0 \)), so we cannot use Theorem 12.3. Let us investigate Theorem 12.4: The area where \( V_2(x) f(x) = 0 \) is \( x_2 = 0 \) and \( x_1 \) arbitrary. However, that means that \( \dot{x}_2 = -\frac{1}{m} (k_0 (x_1 - s_0) + k_1 (x_1 - s_0)^3) < 0 \) if \( x_1 \neq s_0 \) (since \( k_0, k_1 > 0 \)), so no trajectory (except the equilibrium point \( x_1 = s_0 \)) remains in this area. We can thus conclude that \( x_1 = s_0, x_2 = 0 \) is a globally asymptotically stable equilibrium point, according to Theorem 12.4.

3. (a) It can be seen that the linear block of the system consists of a pure integrator and two first-order filters. Assume that the system output is positive. Then the input of the relay is negative due to the feedback and the output of the relay is negative and constant. After integration, it becomes a negative ramp and drives the filters’ output to zero. When the system output (the filters’ output) becomes negative, the input of the integrator flips to a positive constant and the process repeats itself in the opposite direction.
(b) 

\[
\text{Re } G(i\omega) = \frac{-k(T_1 + T_2)}{(T_1^2 \omega^2 + 1)(T_2^2 \omega^2 + 1)}, \quad \text{Im } G(i\omega) = \frac{k(T_1 T_2 \omega^2 - 1)}{\omega(T_1^2 \omega^2 + 1)(T_2^2 \omega^2 + 1)}.
\]

Since \( N(C) \) is real, the imaginary part of \( G(i\omega) \) has to be zero, for \( N(C)G(i\omega) \) to be real. Thus \( \omega = \frac{1}{\sqrt{T_1 T_2}} \).

Further

\[
\text{Re } G(i\omega)|_{\omega=\frac{1}{\sqrt{T_1 T_2}}} = \frac{-kT_1 T_2}{T_1 + T_2}
\]

and the condition for a sine-wave oscillation is

\[
G(s) = -\frac{1}{N(C)}
\]

or

\[
\frac{kT_1 T_2}{T_1 + T_2} = \frac{\pi C}{4b}
\]

and, finally, we have

\[
C = \frac{4b T_1 T_2 k}{\pi(T_1 + T_2)}.
\]

The amplitude of the oscillation clearly increases with the linear block gain \( k \).

4. (a) 

\[
Y(s) = \frac{s + 2.5}{(s + 2.5)(s - 1)}
\]

\[(s^2 + 1.5s - 2.5)Y(s) = (s + 2.5)U(s)\]

\[\ddot{y} + 1.5\dot{y} - 2.5y = \dot{u} + 2.5u\]

State-space realization:

\[x_1 = y, \quad x_2 = \dot{y} - u\]

\[\dot{x} = Ax + Bu\]

\[A = \begin{bmatrix} 0 & 1 \\ 2.5 & -1.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\]

Controllability check:

\[\text{rank } [AB \ B] = \text{rank } \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 1\]

The system realization is not controllable and therefore not minimal.

(b) The transfer function

\[
G(s) = \frac{s + 2.5}{s^2 + 1.5s - 2.5}
\]

is written in canonical observability form as

\[\dot{x} = A_1 \bar{x} + B_1 u,\]

where

\[A_1 = \begin{bmatrix} 0 & 2.5 \\ 1 & -1.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 2.5 \\ 1 \end{bmatrix},\]

and thus it is observable.