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Summary of lecture 6 (I/II)

We can approximate a nonlinear system

\[ \dot{x} = f(x, u), \quad y = h(x, u), \]

by linearizing the system around an equilibrium (stationary) point \((x_0, u_0)\). Intuitively this amounts to approximating the system by a flat hyperplane (straight line in the scalar case).

Let \( \Delta x(t) = x(t) - x_0, \quad \Delta u(t) = u(t) - u_0, \quad \Delta y(t) = y(t) - y_0. \)

A Taylor expansion (only keeping the linear terms) results in

\[
\frac{d}{dt} \Delta x = A \Delta x + B \Delta u, \quad \Delta y = C \Delta x + D \Delta u.
\]
Summary of lecture 6 (II/II)

- **Node**
  
  ![Node Diagram](image)
  
  (Two real e.v. same sign)

- **Saddle**
  
  ![Saddle Diagram](image)
  
  (Two real e.v. different sign)

- **Node and star node**
  
  ![Node and Star Node Diagrams](image)
  
  (Two equal e.v.)

- **Focus and center**
  
  ![Focus and Center Diagrams](image)
  
  (Complex e.v.)
Phase portraits for nonlinear systems

Close to an equilibrium point the dynamics are determined by the linearized dynamics.

- If the linearized system has a center, the nonlinear system has a center or a focus.
- If the linearized system has a node, a focus, or a saddle point, the same applies for the nonlinear system.
- If the linearized system has a star node, several possibilities can occur for the nonlinear system.
Phase portraits far away from equilibrium

Second-order nonlinear system

\[
\dot{x}_1 = f_1(x_1, x_2),
\]
\[
\dot{x}_2 = f_2(x_1, x_2).
\]

Equation for the trajectories ("eliminate time")

\[
\frac{dx_2}{dx_1} = \frac{dx_2}{dt} \frac{dt}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}.
\]

• The trajectories have slope 0 when \( f_2(x_1, x_2) = 0 \).
• The trajectories have slope \( \infty \) when \( f_1(x_1, x_2) = 0 \).
• Study the limits

\[
\lim_{x_1 \to \pm \infty} \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}, \quad \lim_{x_2 \to \pm \infty} \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)},
\]

to find out what the phase portrait looks like far from the origin.
Phase portraits far away from equilibrium

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Phase portraits far away from equilibrium

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\end{align*}
\]

to find out what the phase portrait looks like far from the origin.
Phase portrait of an inverted pendulum

Open loop (autonomous) dynamics

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \sin x_1 - \gamma x_2.
\end{align*}
\]

Left: Nonlinear phase portrait. Right: Phase portrait of the linearized dynamics.
Stability

Given the requirement that the solution must remain within a distance of $\epsilon$ from the equilibrium point $x_0$ for all future $t$, we must be able to find a ball of radius $\delta$ centered in $x_0$, such that the requirement is fulfilled for all starting points $x(0)$ in this ball.
Asymptotic stability

If the system is started sufficiently close to $x_0$ it will eventually end up in $x_0$ as $t \to \infty$. 
Large scale terrain modelling for autonomous mining by Johan Norberg, performed at the University of Sydney, Australia.
Lyapunov functions and stability

**Theorem:** If a Lyapunov function $V$ satisfying

$$V_x(x(t)) f(t) < 0, \ x \neq x_0, \quad V(x) \to \infty \quad \text{as} \quad |x| \to \infty$$

can be found, then the equilibrium point $x_0$ is globally asymptotically stable.

The tricky part is to find the Lyapunov function!
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The tricky part is to **find** the Lyapunov function!
Lyapunov functions for linear systems

**Lemma:** 1. If $A$ has all its eigenvalues in the LHP, then for every matrix $Q = Q^T > 0$ (or $Q \geq 0$), there is a matrix $P = P^T > 0$ (or $P \geq 0$) that satisfies the Lyapunov equation,

$$A^T P + PA = -Q.$$

2. If there are matrices $P = P^T \geq 0$ and $Q = Q^T \geq 0$ satisfying the Lyapunov equation and $(A, Q)$ is detectable, then $A$ has all its eigenvalues strictly in the LHP.

Not very useful in itself, since we already knew this... However, it opens up for more interesting results.
Lyapunov functions for linear systems

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Lyapunov functions for linearized systems

Consider the nonlinear system

$$\dot{x} = Ax + g(x),$$

where $g(x)$ is sufficiently small close to the origin. Let $A$ have eigenvalues in the LHP and let $P$ and $Q$ be positive definite matrices fulfilling

$$A^T P + PA = -Q.$$

Then $V(x) = x^T Px$ is a Lyapunov function in an area around $x = x_0$ also for the nonlinear system.
Globally asymptotically stable: The equilibrium point $x_0$ is globally asymptotically stable (GAS) if it is stable and $x(t) \to x_0, t \to \infty$ for every starting point $x(0)$.

Lyapunov function: A Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ is a generalized distance from $x$ to an equilibrium point $x_0$, satisfying $V(x_0) = 0$, $V(x) > 0$, $x \neq x_0$ and $V_x(x)f(x) \neq 0$.

Lyapunov equation: The Lyapunov equation $A^T P + PA = -Q$ can be used to find Lyapunov functions for linear (and linearized) systems.