Automatic Control III

Lecture 6 – Linearization and phase portraits

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**Summary of lecture 5 (I/III)**

\( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) synthesis:

- Make \( W_u G_{wu}, W_S S, W_T T \) small.
- \( \mathcal{H}_2 \): Minimize \( \int \left( |W_u G_{wu}|^2 + |W_S S|^2 + |W_T T|^2 \right) d\omega \).
- \( \mathcal{H}_\infty \): Set an upper bound for \( |W_u G_{wu}|, |W_S S|, |W_T T| \ \forall \ \omega \).
- Results in algebraic Riccati equations.
$\mathcal{H}_2$, $\mathcal{H}_\infty$ synthesis – pros and cons:

(+ ) Directly handles the specifications on $S, T$ and $G_{wu}$
(+ ) Let us know when certain specifications are impossible to achieve (via $\gamma$).
(+ ) Easy to handle several different specifications (in the frequency domain)
(- ) Can be hard to control the behaviour in the time domain in detail.
(- ) Often results in complex controllers (number of states in the controller = number of states in $G, W_u, W_S, W_T$).
Summary of lecture 5 (III/III)

Linear multivariable controller synthesis summary:

1. Perform an RGA analysis
2. Use simple SISO controllers of PID type if the RGA analysis indicates that it might be possible.
3. Otherwise make use of LQ, MPC or $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis.
DC motor – saturated control signal (I/V)

- DC motor controlled using a lead controller.
- We want to control the motor angle.
- The saturation

\[
    u = \text{sat}(\tilde{u}) = \begin{cases} 
        \tilde{u} & |\tilde{u}| \leq 1, \\
        1 & \tilde{u} > 1, \\
        -1 & \tilde{u} < -1, 
    \end{cases}
\]

renders the system nonlinear.
DC motor – saturated control signal (II/V)

Step responses for two different amplitudes of the reference \( r \) signal.

**Blue:** Amplitude 1  
**Red:** Amplitude 5 (scaled with \( 1/5 \))

**Conclusion:** The step response is amplitude dependent. If the system would have been linear the two step responses would have coincided.
Step responses for two different amplitudes of the reference $r$ signal.

**Blue:** Amplitude 1  
**Red:** Amplitude 5 (scaled with 1/5)

**Conclusion:** The step response is **amplitude dependent**. If the system would have been linear the two step responses would have coincided.
Both the ramp and the sine responses are roughly the same as for a linear system.
DC motor – saturated control signal (IV/V)

Red: \( r \). Blue: \( y \). Green: \( y \) when \( r \) is a ramp (same as on the previous slide).

Something happens here: There is no sine present in the response and the ramp error has increased...
This violates the superposition principle and the frequency fidelity!
Red: \( r \). Blue: \( y \). Green: \( y \) when \( r \) is a ramp (same as on the previous slide).

Something happens here: There is no sine present in the response and the ramp error has increased...
This violates the superposition principle and the frequency fidelity!
Red: before the saturation ($\tilde{u}$). Blue: after the saturation ($u$).

DC motor – saturated control signal (V/V)
We can approximate a nonlinear system
\[ \dot{x} = f(x, u), \quad y = h(x, u), \]
by linearizing the system around an equilibrium (stationary) point \((x_0, u_0)\). Intuitively this amounts to approximating the system by a flat hyperplane (straight line in the scalar case).

Let \( \Delta x(t) = x(t) - x_0, \quad \Delta u(t) = u(t) - u_0, \quad \Delta y(t) = y(t) - y_0. \)

A Taylor expansion (only keeping the linear terms) results in
\[
\frac{d}{dt} \Delta x = \begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta u \end{pmatrix} \quad \Delta y = \begin{pmatrix} C \\ D \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta u \end{pmatrix},
\]
\[ \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \frac{\partial f(x_0, u_0)}{\partial x} \\ \frac{\partial f(x_0, u_0)}{\partial u} \end{pmatrix}, \quad \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} \frac{\partial h(x_0, u_0)}{\partial x} \\ \frac{\partial h(x_0, u_0)}{\partial u} \end{pmatrix}. \]
Phase portraits for linear systems

- **Sign:** Is the solution moving towards the origin or away from the origin (along the eigenvector)?

- **Relative size:** “fast” and “slow” eigenvectors, which is dominating the solution behaviour for $t \approx 0$ and $t \gg 0$?

- **Complex/real:** Complex conjugated eigenvalues results in circles and spirals.

Cases to consider:
1. Two distinct real-valued eigenvalues imply two eigenvectors.
2. Multiple eigenvalues.
3. Complex eigenvalues.
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Cases to consider:

1. Two distinct real-valued eigenvalues imply two eigenvectors.
2. Multiple eigenvalues.
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Two distinct eigenvalues with the same sign

The solution is \( x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 \).

**Stable node:** For eigenvalues \( \lambda_1 < \lambda_2 < 0 \). The first term dominates for small \( t \), the second term dominates for large \( t \).

**Unstable node:** For eigenvalues \( 0 < \lambda_1 < \lambda_2 \). Also here, the first term dominates for small \( t \), the second term dominates for large \( t \).
Two distinct eigenvalues with the same sign

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Two distinct eigenvalues with opposite sign

For eigenvalues $\lambda_1 < 0 < \lambda_2$ (with corresponding eigenvectors $v_1, v_2$) the solution is

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2.$$ 

Trajectories close to $v_1$ will approach the origin. $v_1$ is called the **stable eigenvector**.

Trajectories close to $v_2$ will move away from the origin. $v_2$ is called the **unstable eigenvector**.

Saddle point
A multiple eigenvalue

For multiple eigenvalues $\lambda_1 = \lambda_2$.

Stable node (unstable: change direction).

Stable star node (unstable: change direction).
Two examples in 3D

Example of a generalization to 3D.

Left: Focus + one real eigenvalue.
Right: Three real eigenvalues.
A few concepts to summarize lecture 6

**Equilibrium points:** An equilibrium point is a point $x_0, u_0$ where the system is in rest, i.e. $f(x_0, u_0) = 0$. Also referred to as stationary points.

**Linearization:** Find a Taylor expansion of the nonlinear system around an equilibrium point and only keep the linear parts. This means that we are approximating the system using a flat hyperplane.

**Phase plane:** A two dimensional state space that is simple to visualize graphically.

**Phase portraits:** A plot where one state variable is plotted against another state variable.

**Limit cycle:** A limit cycle is a closed trajectory in phase space having the property that at least one other trajectory spirals into it either as time approaches $\pm$ infinity.