Automatic Control III

Lecture 3 – Basic limitations and conflicts

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Find the poles and the zeros directly from $G(s)$ for a MIMO system:

- The pole polynomial is the least common denominator of all minors of $G(s)$. The system poles are then given by the zeros of the pole polynomial.
- The zero polynomial is the greatest common divisor of the numerators of the maximal minors (after they have been normalized to have the pole polynomial as their denominator).
Summary of lecture 2 (II/II)

Important transfer functions:

\[ G_c = (I + GF_y)^{-1}GF_r \]
\[ S = (I + GF_y)^{-1} \]
\[ S_u = (I + F_y G)^{-1} \]
\[ T = (I + GF_y)^{-1}GF_y \]

Relations among signals:

\[ z = G_c r + Sw - Tn + GS_u w_u \]
\[ u = S_u F_r r - S_u F_y (w + n) + S_u w_u \]

Stability of \( S, S_u, G_{wu}, G_{wuy} \) (and \( F_r \)) guarantees internal stability of the closed-loop system.
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Stability of \( S, S_u, G_{wu}, G_{wy} \) (and \( F_r \)) guarantees internal stability of the closed-loop system.
Compromise $S$ and $T$ – bounding loop gain

Ideally, the sensitivity functions $S$ and $T$ should both be small, but

$$S + T = 1.$$

Compromise:

- Let $S$ be small for low frequencies (dampen process disturbances, insensitivity to modeling errors)
- Let $T$ be small for high frequencies (reduce the effect of measurement errors, stability)

Note that both $S$ and $T$ are uniquely determined by the loop gain $GF_y$. 
Compromise $S$ and $T$ – bounding loop gain

For a small $\epsilon > 0$ we have the following approximate relationships

\[
|S| < \epsilon \iff |GF_y| > \frac{1}{\epsilon},
\]

\[
|T| < \epsilon \iff |GF_y| < \epsilon.
\]

This is another way of seeing that $S$ and $T$ can not be made “small” at the same frequencies.

Relevant question: How fast can we transition from “a small $S$” to “a small $T$”? 
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Relevant question: How fast can we transition from “a small $S$” to “a small $T$”? 
How fast can $|GF_y|$ change?

How small can $\omega_1 - \omega_0$ become?

There is a relationship between the amplitude and the phase of transfer functions (e.g. $GF_y$) that prevents us from making an arbitrarily fast transition...
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There is a relationship between the amplitude and the phase of transfer functions (e.g. $GF_y$) that prevents us from making an arbitrarily fast transition...
Bode’s relation – coupling of amplitude and phase

**Theorem:** Let \( f(x) = \log |G(ie^x)| \). We have

\[
\arg G(i\omega) \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{df}{dx} f(x) \cdot \psi(x - \log \omega) \, dx
\]

where the weight function \( \psi \) is given by

\[
\psi(x) = \log \frac{e^x + 1}{|e^x - 1|}.
\]

(The inequality is replaced by an equality if \( G(s) \) do not have any zeros in the RHP.)

**Interpretation:** Bode’s relationship provides an upper bound on the phase, which depends on the derivative of the amplitude curve.
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Interpretation: Bode’s relationship provides an upper bound on the phase, which depends on the derivative of the amplitude curve.
How small can $S$ become? – an example

$$G(s) = \frac{1}{s^2 + s + 1}, \quad F_y(s) = K.$$  

The frequency scale is logarithmic to the left and linear to the right.

This means even if we neglect the fact that $S + T = 1$, we cannot make $S$ arbitrarily small everywhere!
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How small can $S$ become? – Bode’s integral theorem

Assume that the loop gain $GF_y$ has $M$ poles in the RHP: $p_i; i = 1,\ldots, M$ and that $|GF_y|$ decays at least as $|s|^{-2}$ when $|s| \to \infty$. Then, (scalar case)

$$
\int_{0}^{\infty} \log |S(i\omega)| d\omega = \pi \sum_{i=1}^{M} \text{Re}(p_i).
$$

Multivariable:

$$
\int_{0}^{\infty} \log |\det S(i\omega)| d\omega = \pi \sum_{i=1}^{M} \text{Re}(p_i).
$$

where $|\det S| = \sigma_1 \cdots \sigma_m$. 
A sensitivity $|S(i\omega)| < 1$ for certain frequencies (red region) has to be payed back with $|S(i\omega)| > 1$ for other frequencies (blue region). If $GF_y$ is unstable the situation gets even worse.
The waterbed effect


T. Schön, 2016 Automatic Control III, Lecture 3 – Basic limitations and conflicts
1. **Stable systems**: The sensitivity cannot be $< 1$ for all frequencies, \( \int_0^\infty \log |S(i\omega)| \, d\omega = 0 \).

2. \( \int_0^\infty \log |S(i\omega)| \, d\omega \) is **invariant** to the choice of the controller.

3. The assumption on decay rate ($|s|^{-2}$ for large $s$) is fulfilled if both $G(s)$ and $F_y(s)$ are strictly proper (physically reasonable).

4. For unstable loop gains the situations gets worse. The regions where $|S(i\omega)| > 1$ dominate. The faster the unstable poles, the worse the situation becomes.
Controlling unstable systems

• Requires very reliable controllers! If it breaks...


• A limited control signal implies that stabilization is normally only possible in part of the state space.

• An unstable real pole $p_1$ sets a lower bound for the bandwidth:

$$\omega_B > 2p_1 \text{ (roughly)}$$
Balancing a stick (I/III)

Consider the task of balancing a stick of length $l$ and mass $m$ on your finger.

Input (the acceleration of the finger): $u = \ddot{x}$.
Output (stick angle): $y = \varphi$.

Position of the center of mass:

$$
\left( x + \frac{l}{2} \sin(\varphi), \quad \frac{l}{2} \cos(\varphi) \right).
$$

Newton tells us (in the $x$ direction):

$$
F \sin(\varphi) = m \ddot{x} + m \frac{d^2}{dt^2} \left( \frac{l}{2} \sin(\varphi) \right)
= m \ddot{u} + m \frac{l}{2} (\dot{\varphi} \cos(\varphi) + \dot{\varphi}^2 \sin(\varphi)).
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Balancing a stick (II/III)

Newton tells us (in the $y$ direction):

$$F \cos(\varphi) - mg = m \frac{d^2}{dt^2} \left( \frac{l}{2} \cos(\varphi) \right) = m \frac{l}{2} \left( -\ddot{\varphi} \sin(\varphi) - \dot{\varphi}^2 \cos(\varphi) \right).$$

Combining these two equations will now result in

$$\frac{l}{2} \ddot{\varphi} - g \sin(\varphi) = -u \cos(\varphi).$$

Assume that $\varphi$ is small (i.e. $\sin(\varphi) \approx \varphi$, $\cos(\varphi) \approx 1$) results in

$$G(s) = \frac{-2/l}{s^2 - \frac{2g}{l}}.$$
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\[ G(s) = \frac{-2/l}{s^2 - \frac{2g}{l}} \]
We have now shown that the “stick-system” is unstable with poles in

\[ \pm \sqrt{\frac{2g}{l}} \]

Assuming that we have a \( \tau = 0.1 \text{ s} \) delay in our eye-hand coordination (0.1 s \( \Leftrightarrow \) 10 Hz)

\[ \omega_B < \frac{1}{\tau}; \quad \omega_B = 2\pi 10 \approx 2\pi \sqrt{\frac{2g}{l}} \Rightarrow l \approx 20\text{ cm} \]

\( l = 1 \text{ m} \Rightarrow \omega_B \approx 4.4 \text{ Hz.} \)

If you do not want to balance the stick yourself you can of course have someone (or rather something) else balance it for you:

https://www.youtube.com/watch?v=XxFZ-VStApo
Yet another example of an unstable system

The transfer function for an ordinary bike (from steering angle to tilt angle):

\[
\text{const} \cdot V \frac{s + V/a}{s^2 - g/h}
\]

\(V\): speed, \(h\): height of the center of masses (CoM), \(a\): distance CoM to the rear wheel, \(g = 9.82\) m/s\(^2\).

- Similar to an inverted pendulum.
- Pole in the RHP
  - Depends on the height of the bike: A low bike is harder to balance than a tall bike (the unstable pole is moved further into the RHP)
- Zero in the LHP
  - Depends on the speed and the location of the CoM.
Zeros in the RHP – example

Step response for

\[
\frac{-4s + 2}{(s + 1)(s + 2)}
\]

“The step response initially heads in the wrong direction”
Zeros in the RHP – intuition

- “The system gain for fast changes has the opposite sign compared to changes with slow changes.”
- “A controller that controls slow changes well makes use of the wrong sign for fast changes and could thereby destabilize the system.”
- “Avoid fast changes by giving the system a low bandwidth.”
- **Conclusion:** A zero $z_0$ in the RHP appears to impose an upper limit on the bandwidth.

More rigorously we can make use of Theorem 7.4 to obtain the following rule of thumb:

$$\omega_B \leq \frac{z_0}{2}$$
Zeros and feedback

Consider

\[ G_c = \frac{GF_r}{1 + GF_y}, \quad T = \frac{GF_y}{1 + GF_y} \]

If \( G(z_0) = 0 \), then we also have \( G_c(z_0) = 0 \) and \( T(z_0) = 0 \).

- The zeros of \( G(s) \) will also become the zeros of \( G_c(s) \) and \( T(s) \), i.e. feedback cannot move the zeros.
- Zeros can sometimes be cancelled.
- But, a zero in the RHP cannot be cancelled, since that would require an unstable pole in \( F_y \) and/or \( F_r \).
- Hence, it is impossible to get around the limitations imposed by zeros in the RHP.
Both a pole and a zero in the RHP – example

Transfer function for a bicycle with rear wheel steering:

\[ \text{const} \cdot V \frac{s + V/a}{s^2 - g/h} \]

If the pole is to the right of the zero the systems is extremely hard to control.

“Our rules of thumb for the crossover frequency collide.”

Dynamics and control of bicycles:

Both a pole and a zero in the RHP – example

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Dynamics and control of bicycles:

Poles and zeros in the RHP – summary

1. Pole in the RHP
   • Implies a **lower** limit on the bandwidth (fast control)
   • High operational reliability requirements (recall Gripen accident)
   • Harder when the pole is moved to the **right**

2. Zero in the RHP
   • Implies an **upper** limit on the bandwidth (slow control)
   • Harder when the zero is moved to the **left**

3. Both a zero and a pole in the RHP
   • Zero to the right of the pole: Can be ok.
   • Zero to the left of the pole: extremely hard to control.

On 24 November 2004, the passenger ferry Casino Express was grounded while entering the port of Umeå due to high winds.

Crash investigation:

- The wind power on the upper parts was at least 600 kN (20 m/s wind speed).
- No combination of control signals (propellers, rudders) could have compensated this.
- Not even tug assistance (up to 260 kN) was enough.
Control signal – compensate for disturbances

Control signal: \( u \), disturbance signal: \( d \)

\[
u = G(s)u + G_d(s)d,
\]

for some \( G, G_d \) (scalar for simplicity)

- Assume that \( |u(t)| \leq u_0 \) and \( |d(t)| \leq d_0 \).
- Then it must hold that

\[
u_0 \geq \frac{|G_d(i\omega)|}{|G(i\omega)|} d_0, \quad \forall \omega
\]

if \( d \) is to be perfectly eliminated.

- If this is not fulfilled, there is no controller (linear or nonlinear) that can provide perfect disturbance attenuation.
A few concepts to summarize Lecture 3

**Bode’s relation:** reveals a fundamental coupling between the phase and the amplitude. More specifically, it provides an upper bound on the phase, which depends on the derivative of the amplitude curve.

**Bode’s integral:** Provides a fundamental limitation in terms of what can be achieved by control. Views control design as a way of redistributing disturbance attenuation over different frequencies.

**Waterbed effect:** If disturbance attenuation is improved in one frequency range, it will be worse in another.

**Poles in RHP:** Impose a lower limit on the bandwidth and it impose very high operational reliability requirements. Harder when the pole is moved to the right.

**Zeros in RHP:** Impose an upper limit on the bandwidth. Harder when the zero is moved to the left.