

Kvantfysik Lecture Notes No. 13

In this lecture we will discuss quantum mechanics for more than one particle.

1 Quantum mechanics for more than one particle

Consider the classical energy for N particles with a potential that depends on their positions:

$$E = \sum_{j=1}^N \frac{\vec{p}_j \cdot \vec{p}_j}{2m_j} + V(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N).$$

This suggests that the Schrödinger equation for such a setup is

$$i\hbar \frac{\partial}{\partial t} \Psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N; t) = -\hbar^2 \sum_{j=1}^n \frac{1}{2m_j} \vec{\nabla}_j^2 \Psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N; t) + V(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) \Psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N; t),$$

where now the wave function Ψ depends on the positions of N particles. The Hilbert space is the square integrable functions over the integrals of the N positions, satisfying

$$\int \prod_{j=1}^n d^3r_j \Psi^*(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N; t) \Psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N; t) < \infty.$$

After normalizing, $\Psi^* \Psi$ gives the probability density that the first particle is at \vec{r}_1 and the second one is at \vec{r}_2 and etc. Naturally, we also have the time independent Schrödinger equation

$$E\psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) = -\hbar^2 \sum_{j=1}^n \frac{1}{2m_j} \vec{\nabla}_j^2 \psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) + V(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) \psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N).$$

For now we will limit our discussion to two general types of potentials

- Two particles with $V(\vec{r}_1, \vec{r}_2) = V(\vec{r}_1 - \vec{r}_2)$

- N particles with central potentials: $V(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) = \sum_{j=1}^N U_j(\vec{r}_j)$.

In the first case, since the potential only depends on the difference in positions, it is convenient to choose new variables

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \quad \vec{r} = \vec{r}_1 - \vec{r}_2.$$

The variable \vec{R} is the position of the center of mass, while \vec{r} is the separation between the two particles. In terms of these new variables, we have that

$$\begin{aligned} \vec{\nabla}_1 &= (\vec{\nabla}_1 \vec{R}) \cdot \vec{\nabla}_R + (\vec{\nabla}_1 \vec{r}) \cdot \vec{\nabla}_r = \frac{m_1}{m_1 + m_2} \vec{\nabla}_R + \vec{\nabla}_r \\ \vec{\nabla}_2 &= (\vec{\nabla}_2 \vec{R}) \cdot \vec{\nabla}_R + (\vec{\nabla}_2 \vec{r}) \cdot \vec{\nabla}_r = \frac{m_2}{m_1 + m_2} \vec{\nabla}_R - \vec{\nabla}_r \end{aligned}$$

Hence, we have for the combination in the Hamiltonian

$$\begin{aligned} -\frac{\hbar^2}{2m_1} \vec{\nabla}_1^2 - \frac{\hbar^2}{2m_2} \vec{\nabla}_2^2 &= -\frac{\hbar^2}{2} \left(\frac{m_1}{(m_1 + m_2)^2} \vec{\nabla}_R^2 + \frac{1}{m_1} \vec{\nabla}_r^2 + \frac{1}{m_1 + m_2} \vec{\nabla}_r \cdot \vec{\nabla}_R \right. \\ &\quad \left. + \frac{m_2}{(m_1 + m_2)^2} \vec{\nabla}_R^2 + \frac{1}{m_2} \vec{\nabla}_r^2 - \frac{1}{m_1 + m_2} \vec{\nabla}_r \cdot \vec{\nabla}_R \right) \\ &= -\frac{\hbar^2}{2(m_1 + m_2)} \vec{\nabla}_R^2 - \frac{\hbar^2(m_2 + m_1)}{2m_1 m_2} \vec{\nabla}_r^2 = -\frac{\hbar^2}{2M} \vec{\nabla}_R^2 - \frac{\hbar^2}{2\mu} \vec{\nabla}_r^2, \end{aligned}$$

where $M = m_1 + m_2$ is the total mass and $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass. The variables are now separated so we can reduce the time independent Schrödinger equation to an equation for each variable. The wave-function can be written as a product, $\psi(\vec{R}, \vec{r}) = \psi_R(\vec{R}) \psi_r(\vec{r})$, and each function in the product satisfies a Schrödinger equation

$$\begin{aligned} E_R \psi_R(\vec{R}) &= -\frac{\hbar^2}{2M} \vec{\nabla}_R^2 \psi_R(\vec{R}) \\ E_r \psi_r(\vec{r}) &= -\frac{\hbar^2}{2\mu} \vec{\nabla}_r^2 \psi_r(\vec{r}) + V(\vec{r}) \psi_r(\vec{r}), \end{aligned}$$

with the total energy E given by $E = E_R + E_r$. For the center of mass motion, we see that

$$\psi_R(\vec{R}) = A e^{i\vec{K} \cdot \vec{R}}, \quad E_R = \frac{\hbar^2 \vec{K} \cdot \vec{K}}{2M},$$

where $\hbar \vec{K}$ is the center of mass momentum. And so the total energy is the kinetic energy from the center of mass motion plus the contribution due to the interaction between the particles, which is the same situation as in classical physics.

In the second case with N particles seeing central potentials, the variables are already separated so we can immediately proceed to a solution of the Schrödinger equation, namely

$$\psi(\vec{r}_1, \vec{r}_2 \dots, \vec{r}_N) = \psi_1(\vec{r}_1)\psi_2(\vec{r}_2) \dots \psi_N(\vec{r}_N) \quad (1)$$

where each *single-particle wave-function* $\psi_j(\vec{r}_j)$ is a solution to a single-particle Schrödinger equation

$$E_j\psi_j(\vec{r}_j) = -\frac{\hbar^2}{2m_j}\vec{\nabla}_j^2\psi_j(\vec{r}_j) + U_j(\vec{r}_j)$$

and with total energy $E = \sum_{j=1}^N E_j$. Now it is possible that different combinations of the E_j give the same energy E , in which case the wave function could be a sum of the product terms in (1).

Let us now assume that the particles all have the same mass, $m_j = m$, and have the same central potential $U_j(\vec{r}_j) = U(\vec{r}_j)$. Then clearly there are many degeneracies, since all particles have the same set of single-particle wave-functions as solutions. Let us call the normalized single-particle eigenfunctions $\psi_k(\vec{r})$ and the single-particle energies E_k , with k labeling which single-particle state the particle is in. In other words, these are solutions to the single-particle time independent Schrödinger equation

$$E_k\psi_k(\vec{r}) = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi_k(\vec{r}) + U(\vec{r})\psi_k(\vec{r}).$$

Then a solution of the full time independent Schrödinger equation is

$$\psi(\vec{r}_1, \vec{r}_2 \dots, \vec{r}_N) = \psi_{k_1}(\vec{r}_1)\psi_{k_2}(\vec{r}_2) \dots \psi_{k_N}(\vec{r}_N)$$

and the energy is $E = E_{k_1} + E_{k_2} + \dots + E_{k_N}$. But we can find another solution with the same energy by switching the single-particle states around, so for example

$$\psi(\vec{r}_1, \vec{r}_2 \dots, \vec{r}_N) = \psi_{k_2}(\vec{r}_1)\psi_{k_1}(\vec{r}_2) \dots \psi_{k_N}(\vec{r}_N)$$

has the same energy. Clearly we can switch the other single-particle states around as well. If all k_j are different, then this gives us $N!$ states with the same energy, and any linear combination of these states still satisfies the time independent Schrödinger equation. This can be a lot of states! Remarkably, in nature it turns out that there are not as many.

2 Identical Particles

Suppose we have a many particle wave function $\psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N)$. Let us define the “exchange operator” P_{ij} which exchanges the positions of the i^{th} and j^{th} particles. So for example, $P_{12}\psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) = \psi(\vec{r}_2, \vec{r}_1, \dots, \vec{r}_N)$. Let us suppose that the particles are identical, which means that there is no experiment that distinguishes any one particle from any other (for the moment we are assuming that the particles do not have spin or any other “internal” quantum numbers.) This means that probability density *cannot* change if we exchange the positions of two particles in the wave-function, otherwise there *would* be an experiment to distinguish the particles – put one particle at \vec{r}_1 and another at \vec{r}_2 and measure the probability, and then reverse the order. If the probability changes we can distinguish the particles by what probability we find. Hence if the particles really are identical then

$$P_{12}\psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) = \psi(\vec{r}_2, \vec{r}_1, \dots, \vec{r}_N) = e^{i\alpha}\psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N),$$

where α is real so $e^{i\alpha}$ is just a phase, leaving the probability density unchanged. Thus we see that ψ is an eigenfunction of P_{12} with eigenvalue $e^{i\alpha}$. But if we exchange \vec{r}_1 and \vec{r}_2 and then do it again, it is the same as doing nothing. So $P_{12}P_{12} = 1$. This means that $(e^{i\alpha})^2 = 1$, so $e^{i\alpha} = \pm 1$.

As it turns out, there are two types of particles in nature, bosons and fermions. Bosons are symmetric under interchange of the particles (the eigenvalue for the exchange operator is $+1$), while fermions are *antisymmetric* under the exchange (their eigenvalue is -1). Bosons all have integer spins and fermions all have half-integer spins. So photons (spin 1) and pions (spin 0) are bosons while electrons, protons and neutrons (spin $1/2$) are fermions (this statement about “spin and statistics” is usually proven in a first course in quantum field theory. For us, we will treat it as a given.)

To simplify the discussion, let us assume that we have two identical particles and that they are in a common central potential with normalized single-particle eigenfunctions $\psi_1(\vec{r})$ and $\psi_2(\vec{r})$. We now wish to construct an eigenfunction which is an eigenstate of the exchange P_{12} . It's not hard to see that the eigenstates are

$$\psi(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}} \left(\psi_1(\vec{r}_1)\psi_2(\vec{r}_2) \pm \psi_2(\vec{r}_1)\psi_1(\vec{r}_2) \right), \quad (2)$$

where the eigenvalue with the $+$ ($-$) sign is $+1$ (-1). Hence, for two identical bosons we choose the $+$ sign and for two identical fermions we choose the $-$ sign. Now looking at this wave-function we learn a very important fact called the *Pauli principle*: **identical bosons can be in the same single**

particle state, while identical fermions cannot. If the fermions were in the same state, then $\psi_1(\vec{r}) = \psi_2(\vec{r})$ and so $\psi(\vec{r}_1, \vec{r}_2) = 0$, in other words there is no state! We also can see something else: if $\vec{r}_1 = \vec{r}_2$ then the wave-function is also zero. Therefore fermions tend to move away from each other. This has profound consequences and is responsible for the structure of atoms, solids, etc. On the other hand, identical bosons have no problem being close to each other.

A brief digression on many identical fermions

The analog of (2) for N identical fermions is the *Slater Determinant*

$$\psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_1(\vec{r}_1) & \psi_2(\vec{r}_1) & \dots & \psi_N(\vec{r}_1) \\ \psi_1(\vec{r}_2) & \psi_2(\vec{r}_2) & \dots & \psi_N(\vec{r}_2) \\ \vdots & \vdots & \dots & \vdots \\ \psi_1(\vec{r}_N) & \psi_2(\vec{r}_N) & \dots & \psi_N(\vec{r}_N) \end{vmatrix}$$

where the straight bracket refers to the determinant of the $N \times N$ matrix. If we were to exchange \vec{r}_j with \vec{r}_k , this has the effect of exchanging the j^{th} and k^{th} rows which changes the sign of the determinant. This also means that the determinant is zero if $\vec{r}_j = \vec{r}_k$. Likewise if $\psi_j = \psi_k$, then the j^{th} and k^{th} columns are the same and the determinant is zero. The factor in front of the determinant is a normalization factor since a determinant is a sum of $N!$ terms, and all the terms are orthogonal to each other, assuming all the ψ_j are orthogonal. Now if the particles were distinguishable, we could consider any coefficient in front of each of the $N!$ terms, giving rise to $N!$ degenerate states. But by being identical, all the coefficients are determined and so instead of $N!$ states there is only one.

Including spin

Up to now we have treated the spin separately from the position of the particle. Now it is time to combine them. We will only consider the case of $s = 1/2$. Let us write the combined state of a single particle as a tensor product of the position state and the spin state. The position state wave-function we write as $\psi(\vec{r})$ and the spin state as $|\chi\rangle$ and we combine these into $\psi(\vec{r})|\chi\rangle$. If we now have two identical fermions then the exchange of the particles not only exchanges their positions, but also their spins. Hence the full antisymmetric wave-function is

$$\frac{1}{\sqrt{2}} \left(\psi_1(\vec{r}_1)|\chi_1\rangle \otimes \psi_2(\vec{r}_2)|\chi_2\rangle - \psi_2(\vec{r}_1)|\chi_2\rangle \otimes \psi_1(\vec{r}_2)|\chi_1\rangle \right),$$

where the symbol \otimes refers to the tensor product of the two dimensional spin

states. The tensor product is $2 \times 2 = 4$ dimensional. For example,

$$\left(a_1|\uparrow\rangle + b_1|\downarrow\rangle\right) \otimes \left(a_2|\uparrow\rangle + b_2|\downarrow\rangle\right) = a_1a_2|\uparrow\uparrow\rangle + a_1b_2|\uparrow\downarrow\rangle + a_2b_1|\downarrow\uparrow\rangle + b_1b_2|\downarrow\downarrow\rangle.$$

Now that there is spin, it is possible for the two particles to be in the same position state. If we let $\psi_1 = \psi_2 = \psi$ then our two particle state becomes

$$\frac{1}{\sqrt{2}} \left(|\chi_1\rangle \otimes |\chi_2\rangle - |\chi_2\rangle \otimes |\chi_1\rangle \right) \psi(\vec{r}_1) \psi(\vec{r}_2) = \frac{a_1b_2 - a_2b_1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \psi(\vec{r}_1) \psi(\vec{r}_2).$$

Clearly, this state is symmetric under the exchange of the positions, but antisymmetric under the exchange of the spins, so the full wave-function is antisymmetric. Furthermore, we recognize the combination of spins as the total spin zero state. In order for this state to be normalized we should select $|a_1b_2 - a_2b_1| = 1$. One way to do this is to choose $a_1 = b_2 = 1$, $a_2 = b_1 = 0$, that is we choose the first spin as up in the z direction and the second as down in z , and then subtract the state with the spins reversed. But we could have also chosen $a_1 = b_1 = a_2 = -b_2 = \frac{1}{\sqrt{2}}$ which corresponds to the first spin being up in the x direction and the second being down and then subtract the reverse. Thus, if one particle is spin up with respect to a particular direction, then the other must be down with respect to the same direction. Of course, no matter how we choose the direction we end up with the same state, as we must since total spin 0 has only one state.

In any case, we now see that it is possible to put two identical spin-1/2 particles into the same position state. But we cannot put any more, because if all particles are in the same position state, then the n -particle state must be completely antisymmetric in the spin. This would mean that all the spin components are different, but this is not possible since there are only two such components.

It is also possible to have a state which is completely symmetric in the spins. For two particles this would have total spin one (for N particles this would be $N/2$.) In this case the wave-function would have to be completely antisymmetric under the exchange of positions.

Identical fermions ($s = 1/2$) in a box – Fermi energy

Suppose we have a large number of identical spin-1/2 particles in a box and we want to find the ground state. For simplicity, let us suppose that the dimensions of the box are $L \times L \times L$, although in the final answer we will see that the actual shape of the box does not matter. Then the single particle wavefunctions are

$$\psi_{n_1, n_2, n_3}(\vec{r}) = \frac{2\sqrt{2}}{L^{3/2}} \sin \frac{\pi n_1}{L} x \sin \frac{\pi n_2}{L} y \sin \frac{\pi n_3}{L} z$$

and the single particle energies are

$$E_{n_1, n_2, n_3} = \frac{\hbar^2 \pi^2}{2mL^2} (n_1^2 + n_2^2 + n_3^2) = \frac{\hbar^2 \pi^2}{2mL^2} \vec{n}^2,$$

where the n_i are integers with $n_i \geq 1$. Each single particle state, that is each choice of \vec{n} , can have two fermions, and to find the ground state we put each fermion into the lowest available state. Now the lowest available state is the open state with the lowest value of \vec{n}^2 . So the first two fermions go into the state $\vec{n} = (1, 1, 1)$, the next six go into $\vec{n} = (2, 1, 1)$, $(1, 2, 1)$ and $(1, 1, 2)$, *etc.* If the number of fermions is very large, a typical number being $N \sim 10^{23}$, then we can make a very useful approximation for filling states and finding the total energy. If we fill the states with N particles then \vec{n}^2 is bounded by a maximal value n_{\max}^2 that depends on the total number of particles. To a very good approximation, we have

$$2 \sum_{\substack{\vec{n}^2 \leq n_{\max}^2 \\ n_i \geq 1}} = N,$$

where the sum is over all positive n_1 , n_2 and n_3 that satisfy the bound $\vec{n}^2 \leq n_{\max}^2$, and the factor of 2 is counting the two spins we can put into each state. If N is very large, then we can approximate the sum by an integral,

$$2 \int_0^{n_{\max}} dn_3 \int_0^{\sqrt{n_{\max}^2 - n_3^2}} dn_2 \int_0^{\sqrt{n_{\max}^2 - n_3^2 - n_2^2}} dn_1 = N.$$

These are not the best coordinates, so let's choose spherical coordinates, $(n_1, n_2, n_3) = (n \sin \theta \cos \phi, n \sin \theta \sin \phi, n \cos \theta)$. Since we are limiting to positive n , the angles are bounded by $0 \leq \theta \leq \pi/2$, $0 \leq \phi \leq \pi/2$. With these coordinates the integral becomes

$$2 \int_0^{n_{\max}} n^2 dn \int_0^{\pi/2} \sin \theta d\theta \int_0^{\pi/2} d\phi = 2 \frac{1}{8} \frac{4\pi}{3} n_{\max}^3 = \frac{\pi}{3} n_{\max}^3 = N.$$

The integral is the volume of one octant of a sphere of radius n_{\max} , multiplied by 2. Expressing n_{\max} in terms of N , we have

$$n_{\max} = \left(\frac{3N}{\pi} \right)^{1/3}.$$

For this value of n_{\max} , the highest energy particles have energy

$$E_F = \frac{\hbar^2 \pi^2}{2mL^2} n_{\max}^2 = \frac{\hbar^2}{2m} \left(\frac{3N\pi^2}{L^3} \right)^{2/3} = \frac{\hbar^2}{2m} \left(\frac{3N\pi^2}{V} \right)^{2/3},$$

where E_F is called the *Fermi energy*. For the ground state, all single-particle states with energies below E_F are occupied (also called “filled”), and all single-particle states with energies greater than E_F are unoccupied (also called “empty”.) For a general shape box, E_F will have the same dependence on the volume. In fact we see that E_F can be rewritten as

$$E_F = \frac{\hbar^2}{2m} (3\rho\pi^2)^{2/3},$$

where $\rho = N/V$ is the density of the particles. If we had chosen a shape other than a cubic box we would have found the same dependence of the Fermi energy on the density ρ .

The total energy E is

$$E = 2 \sum_{(n_1, n_2, n_3)} E_{n_1, n_2, n_3} \approx 2 \int_0^{n_{\max}} n^2 dn \int_0^{\pi/2} \sin \theta d\theta \int_0^{\pi/2} d\phi \frac{\hbar^2 \pi^2}{2mL^2} n^2 = 2 \cdot \frac{\pi}{2} \cdot \frac{\hbar^2 \pi^2}{2mL^2} \cdot \frac{1}{5} n_{\max}^5$$

which we can rewrite as

$$E = \frac{\pi}{5} n_{\max}^3 \frac{\hbar^2 \pi^2}{2mL^2} n_{\max}^2 = \frac{3}{5} N E_F.$$

Suppose we decrease the volume while keeping the number of particles fixed. E will increase because of the volume dependence of E_F . This leads to a *Fermi pressure* (also called a degeneracy pressure),

$$P_F \equiv -\frac{\partial E}{\partial V} = \frac{2}{3} \frac{E}{V} = \frac{2}{5} \rho E_F.$$

To get some idea about how big this is, let us consider electrons with a density $\rho = 6 \times 10^{23}/(\text{cm})^3$. Using $\hbar = 1.05 \times 10^{-27}$ ergs-sec and $m = 9 \times 10^{-28}$ gm, we get

$$P_F = \frac{(1.05 \times 10^{-27} \text{ ergs-sec})^2}{5(9 \times 10^{-28} \text{ gm})} (6 \times 10^{23}/(\text{cm})^3)^{5/3} (3\pi^2)^{2/3} \approx 10^{13} \text{ dyne}/(\text{cm})^2$$

It is this sort of enormous pressure that prevents a solid from collapsing. We can think of this as being due to the “exchange force” between the fermions (see the appendix).

A The exchange force

In this appendix we show that identical bosons seem to be attracted to each other while fermions tend to stay away. First we should say that the exchange

force is something of a misnomer, since the “force” does not come from a attractive or repulsive potential in the Hamiltonian. It is instead a consequence of making the wave-functions symmetric or antisymmetric. To this end, let us first assume that we have two distinguishable particles, but with the same mass and central potential. Then we could have a wave-function $\psi(\vec{r}_1, \vec{r}_2) = \psi_1(\vec{r}_1)\psi_2(\vec{r}_2)$, where $\psi_1(\vec{r})$ and $\psi_2(\vec{r})$ are distinct orthonormal states. Then the expectation value for the separation squared $\langle(\vec{r}_1 - \vec{r}_2)^2\rangle$ is

$$\begin{aligned}\langle(\vec{r}_1 - \vec{r}_2)^2\rangle &= \int d^3r_1 d^3r_2 \psi_1(\vec{r}_1)^* \psi_1(\vec{r}_1) \psi_2(\vec{r}_2)^* \psi_2(\vec{r}_2) (\vec{r}_1^2 + \vec{r}_2^2 - 2\vec{r}_1 \cdot \vec{r}_2) \\ &= \langle 1|\vec{r}^2|1\rangle \langle 2|2\rangle + \langle 1|1\rangle \langle 2|\vec{r}^2|2\rangle - 2\langle 1|\vec{r}|1\rangle \cdot \langle 2|\vec{r}|2\rangle \\ &= \langle 1|\vec{r}^2|1\rangle + \langle 2|\vec{r}^2|2\rangle - 2\langle 1|\vec{r}|1\rangle \cdot \langle 2|\vec{r}|2\rangle \equiv (\Delta\vec{r}_{12})^2,\end{aligned}$$

where

$$\begin{aligned}\langle j|k\rangle &= \int d^3r \psi_j(\vec{r})^* \psi_k(\vec{r}) = \delta_{jk}, \quad \langle j|\vec{r}|k\rangle = \int d^3r \psi_j(\vec{r})^* \vec{r} \psi_k(\vec{r}) \\ \langle j|\vec{r}^2|k\rangle &= \int d^3r \psi_j(\vec{r})^* \vec{r}^2 \psi_k(\vec{r}).\end{aligned}$$

Now let us do the same thing with the wave-function in (2). Here we find

$$\begin{aligned}\langle(\vec{r}_1 - \vec{r}_2)^2\rangle_{\pm} &= \frac{1}{2} \int d^3r_1 d^3r_2 \psi_1(\vec{r}_1)^* \psi_1(\vec{r}_1) \psi_2(\vec{r}_2)^* \psi_2(\vec{r}_2) (\vec{r}_1^2 + \vec{r}_2^2 - 2\vec{r}_1 \cdot \vec{r}_2) \\ &\quad + \frac{1}{2} \int d^3r_1 d^3r_2 \psi_2(\vec{r}_1)^* \psi_2(\vec{r}_1) \psi_1(\vec{r}_2)^* \psi_1(\vec{r}_2) (\vec{r}_1^2 + \vec{r}_2^2 - 2\vec{r}_1 \cdot \vec{r}_2) \\ &\quad \pm 2 \cdot \frac{1}{2} \int d^3r_1 d^3r_2 \psi_1(\vec{r}_1)^* \psi_2(\vec{r}_1) \psi_2(\vec{r}_2)^* \psi_1(\vec{r}_2) (\vec{r}_1^2 + \vec{r}_2^2 - 2\vec{r}_1 \cdot \vec{r}_2) \\ &= \frac{1}{2}(\Delta\vec{r}_{12})^2 + \frac{1}{2}(\Delta\vec{r}_{12})^2 \pm \left(\langle 1|\vec{r}^2|2\rangle \langle 2|1\rangle + \langle 1|2\rangle \langle 2|\vec{r}^2|1\rangle - 2\langle 1|\vec{r}|2\rangle \cdot \langle 2|\vec{r}|1\rangle \right) \\ &= (\Delta\vec{r}_{12})^2 \mp 2|\langle 1|\vec{r}|2\rangle|^2,\end{aligned}$$

where we used that $\langle 1|2\rangle = 0$. Hence, since $|\langle 1|\vec{r}|2\rangle|^2 \geq 0$, bosons tend to move closer together since $\langle(\vec{r}_1 - \vec{r}_2)^2\rangle_+ \leq (\Delta\vec{r}_{12})^2$ while fermions tend to be pushed apart since $\langle(\vec{r}_1 - \vec{r}_2)^2\rangle_- \geq (\Delta\vec{r}_{12})^2$. Of course if the overlap integral $\langle 1|\vec{r}|2\rangle$ is zero then there is no effect.